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Splitting Criteria for Vector Bundles on Singular Quadrics

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Abstract. Here we give a few splitting criteria for vector bundles on singular quadrics.

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1. Introduction

For all integers n, s such that $n \geq 2$ and $-1 \leq s \leq n-2$ let $Q_{n,s} \subset \mathbf{P}^{n+1}$ denote an n-dimensional quadric hypersurface whose singular locus has dimension s. Hence $Q_n := Q_{n,-1}$ is smooth, each $Q_{n,s}$ is integral and for fixed n, s all $Q_{n,s}$'s are projectively normal. We recall the following facts on the smooth quadric hypersurface Q_n ([3], [8], [9], [10], [1]). See Q_{n-1} has a smooth hyperplane section of Q_n If $n = 2m$ is even (resp. $n = 2m-1$ is odd), then there are two non-isomorphic (resp. one) vector bundles S'_n, S''_n (resp. S_n) on Q_n with rank 2^{m-1} such that $S'_{2m}|Q_{2m-1} \cong S''_{2m}|Q_{2m-1} \cong S_{2m-1}$ and $S_{2m+1}|Q_{2m} \cong S'_{2m} \oplus S''_{2m}$. These vector bundles S_n, S'_n and S_n are called the spinor bundles and we will denote them with S_n or just S even if n is not odd. A vector bundle E on Q_n, $n \geq 3$, splits (i.e. it is isomorphic to a direct sum of line bundles if and only if $h^i(Q_n, E(t)) = 0$ for all $t \in \mathbb{Z}$ all $1 \leq i \leq n-1$ and $h^i(Q_n, E(t) \otimes S) = 0$ and all spinor bundles S, all $t \in \mathbb{Z}$ and all $1 \leq i \leq n-2$ ([10], Th. 3.4). When $n = 2$ the same vanishings are equivalent to the splitting of E into a direct some of balanced line bundles, i.e. of line bundles $\mathcal{O}_{Q_2}(t)$ for some $t \in \mathbb{Z}$. If rank(E) is low, say rank$(E) < n$, stronger criteria are true ([6], Th. 2.2 and Th. 3.1). For all integers n, s, t such that $-1 \leq t < s \leq n-2$ see $Q_{n,s}$ has a $(t-s)$-codimensional linear seqction

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of $Q_{n+t-s,t}$. In particular we may see $Q_{n,s}$ as a linear section of the smooth quadric $Q_{n+s+1,-1}$. set $S_{n,s} := S_{n+s+1}|Q_{n,s}$, where S_{n+s+1} denote any spinor bundle of Q_{n+s+1}. Hence $Q_{n,s}$ has one spinor bundle if $n+s+1$ is odd and (a priori) two spinor bundles if $n+s+1$ is even. Here we will prove that the quoted splitting criteria are true also for the singular quadric hypersurfaces $Q_{n,s}$.

Theorem 1. *Fix integers n, s such that $0 \le s \le n-3$. Let E be a vector bundle on $Q_{n,s}$. E is isomorphic to a direct sum of line bundles if and only if $h^i(Q_{n,s}, E(t)) = 0$ for all $t \in \mathbb{Z}$ all $1 \le i \le n-1$ and $h^i(Q_{n,s}, E(t) \otimes S) = 0$ and all spinor bundles S, all $t \in \mathbb{Z}$ and all $1 \le i \le n-2$.*

Proposition 1. *Fix integers n, s such that $0 \le s \le n-7$. There is no rank 2 indecomposable vector bundle E on $Q_{n,s}$ such that $h^i(Q_{n,s}, E(t)) = 0$ for all $1 \le i \le n-2$ and all $t \in \mathbb{Z}$.*

Theorem 2. *Fix integers n, s such that $0 \le s \le n-5$. A vector bundle E on $Q_{n,s}$ with rank$(E) \le n-s-3$ splits if and only if $h^i(Q_{n,s}, E(t)) = 0$ for all $2 \le i \le n-2$ and all $t \in \mathbb{Z}$ and $h^1(Q_{n,s}, E(t) \otimes S) = 0$ for all $t \in \mathbb{Z}$ and all spinor bundles S on $Q_{n,s}$.*

Remark 1. Use induction on s, the corresponding vanishings on Q_{n+s+1} and standard short exact sequences to get $h^i(Q_{n,s}, \mathcal{O}_{Q_{n,s}}(t)) = h^i(Q_{n,s}, S_{n,s}(t)) = 0$ for all $1 \le i \le n-1$ and $h^n(Q_{n,s}, \mathcal{O}_{Q_{n,s}}(t)) = 0$ for $t \ge -n+1$.

Remark 2. Assume $0 \le s \le n-2$. Here we will check that $\mathrm{Pic}(Q_{n,s}) \cong \mathbb{Z}$ and that $\mathcal{O}_{Q_{n,s}}(1)$ is a generator of $\mathrm{Pic}(Q_{n,s})$. Indeed, taking linear sections it is easy to reduce to the case $n=2$ and $s=0$ (the quadric cone in $\mathbf{P}^3$. In this case that the Hirzebruch surface F_2 is the blowing-up of the vertex of $Q_{2,0}$ and that a line of $Q_{n,s}$ is a Weil divisor, but not a Cartier divisor (or use [2], Ex. V. Ex. 2.9, for some help).

Proof of Theorem 1. Remarks 1 and 2 gives the " only if " part. Fix E satisfying all vanishings and let $Q_{n-s-1} \subset Q_{n,s}$ a general codimension $s+1$ linear sections. Since $S_{2m+1}|Q_{2m} \cong S'_{2m} \oplus S''_{2m}$ and $S'_{2m}|Q_{2m-1} \cong |S_{2m-1}$, we get that $S_{n,s}|Q_{n-s-1}$ is isomorphic to a direct sum of spinor bundles of Q_{n-s-1}. A few standard exact sequences and Remark 2 give $h^i(Q_{n-s-1}, (E|Q_{n-s-1})(t)) = 0$ for all $t \in \mathbb{Z}$ all $1 \le i \le n-s-2$ and $h^i(Q_{n,s}, (E|Q_{n-s-1})(t) \otimes S) = 0$ and all spinor bundles S, all $t \in \mathbb{Z}$ and all $1 \le i \le n-s-3$. Since $n-s-1 \ge 2$, [10], Th. 3.4, implies $E|Q_{n-s-1} \cong \oplus_{i=1}^{r} \oplus_{Q_{n-s-1}} (a_i)$, $r := \mathrm{rank}(E)$, for some $a_i \in \mathbb{Z}$. Set $F := \oplus_{i=1}^{r} \oplus_{Q_{n,s}} (a_i)$. Hence $Hom(F, E)$ is isomorphic to a direct sum of copies of twists of E. Hence $h^i(Q_{n-t,t}, Hom(F,E)(t)) = 0$ for all $t \in \mathbb{Z}$ and all $1 \le i \le n-t-1$ for all $0 \le t \le s-1$. The vanishing of $h^1(Q_{n-s,0}, Hom(F,E)(-1))$ gives that any isomorphism $\sigma : E|Q_{n-s-1} \to \oplus_{i=1}^{r} \oplus_{Q_{n-s-1}} (a_i)$ lifts to a morphism $\tau : E|Q_{n-s,0} \to \oplus_{i=1}^{r} \oplus_{Q_{n-s,0}} (a_i)$. τ is a morphism between vector bundles of the same rank whose restriction to each point of an effective ample divisor is an isomorphism. Hence τ is an isomorphism. If $s=0$, then we get $F \cong E$ and hence we are done. If $s > 0$ we use $s - 1$ further steps, each of them equal to the one just done. □

Proof of Proposition 1. Assume the existence of such vector bundle E. See Q_{n-s-1} as a linear section of $Q_{n,s}$. The usual standard exact sequences give $h^i(Q_{n-s-1}, E(t)|Q_{n-s-1}) = 0$ for all $1 \le i \le n - s - 3$ and all $t \in \mathbb{Z}$. Since $n - s - 1 \ge 6$, [6], part 2 of Cor. 3.3, $E|Q_{n-s-1}$ is decomposable. As in the proof of Theorem 1 we get that E is decomposable, contradiction. □

Proof of Theorem 2. Use the proof of Proposition 1 and [6], Th. 2.2. □

It would be nice to extend as much as possible the approach with monads given in [4],[5],[6], [7] and [7] to singular quadrics.

We work over an algebraically closed field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) \neq 2$.

References

[1] V. Ancona and G. Ottaviani, Some applications of Beilinson's theorem to projective spaces and quadrics, Forum Math. 3 (1991), 157–176.

[2] R. Hartshorne, Algebraic Geometry, Springer, Berlin, 1977.

[3] H. Knörrer, Cohen-Macaulay modules of hypersurfaces singularities I, Invent. Math. 88 (1987), 153–164.

[4] N. Mohan Kumar, C. Peterson and A. P. Rao, Monads on projective spaces, Manuscripta Math. 112 (2003), no. 2, 183–189.

[5] N. Mohan Kumar, A. P. Rao and G. V. Ravindra, Arithmetically Cohen-Macaulay bundles on hypersurfaces, axXiv:math.AG/0507161.

[6] F. Malaspina, Monads and vector bundles on quadrics, preprint.

[7] F. Malaspina, Monads and rank 3 vector bundles on quadrics, preprint.

[8] G. Ottaviani, Critéres de scindage pour les fibrés vectoriels sur les grassmanniens et les quadriques, C. R. Acad. Sci. 305 (1987), 257–260.

[9] G. Ottaviani, Spinor bundles on quadrics, Trans. Amer. Math. Soc. 307 (1988), no. 1, 301–316.

[10] G. Ottaviani, Some extensions of Horrocks criterion to vector bundles on Grassmannians and quadrics, Annali Mat. Pura Appl. (IV) 155 (1989), 317–341.

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