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The Defectivity of Zero-Dimensional Subschemes Contained in Double Points

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Abstract. Let $X \subset \mathbf{P}^r$ be an integral projective variety and $Z \subset X$ a general zero-dimensional subscheme for which we prescribe the length of each connected component. Here we study the value of $h^1(\mathbf{P}^r, \mathcal{I}_Z(1))$ when each connected component of Z is contained in a double point of X.

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Let $X \subset \mathbf{P}^N$ be an integral n-dimensional projective variety. For any $P \in X_{reg}$ let $2P$ denote the closed subscheme of X with $(\mathcal{I}_P)^2$ as its ideal sheaf. Hence length$(2P) = n+1$ and $(2P)_{red}$. For every integer i such that $1 \le i \le n+1$ the set $\Lambda(P,i)$ of all zero-dimensional subschemes of $2P$ with length i is parametrized by the Grassmannian of all $(i-1)$-dimensional linear subspaces of T_PX. Hence $\Lambda(P,i)$ is irreducible and of dimension $(i-1)(n-i+1)$. Fix non-negative integers a_i, $1 \le i \le n$, such that $a_1 + \cdots + a_n > 0$. Any such sequence of n integers will be said to be a numerical datum (or a numerical datum for the dimension n) and we will denote it with the symbol $[a_n, \dots, a_1]$. Fix $a_1 + \cdots + a_n$ general points $P_{i,j}$, $1 \le i \le n$, $1 \le j \le a_i$. Let $Z \subset X$ be the zero-dimensional scheme such that $Z_{red} = \{P_{i,j}\}_{1 \le i \le n, 1 \le j \le a_i}$ and the connected component $Z_{i,j}$ of Z supported by $P_{i,j}$ is a general element of $\Lambda(P_{i,j}, i)$. Hence length$(Z) = \sum_{i=1}^{n}(i+1)a_i$. We will say that X is tangentially $[a_n, \dots, a_1]$-defective if $\dim(\langle Z \rangle) < \min\{N, -1 + \sum_{i=1}^{n}(i+1)a_i\}$. We will say that X is minimally tangentially $[a_n, \dots, a_1]$-defective if it is tangentially $[a_n, \dots, a_1]$-defective and there is no n-ple of non-negative integers

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b_i, $1 \le i \le n$, such that $b_i \le a_i$ for all i and X is tangentially $[b_n, \dots, b_1]$-defective. If our algebraically closed base field $\mathbb{K}$ satisfies char$(\mathbb{K}) = 0$, then one could drop the word " tangentially ", because in this case the tangential $[(a_n, 0, \dots, 0]$-defectivity of X is equivalent to the $(a_n - 1)$-defectivity of X in the sense of [2]. Recently, M. C. Brambilla and G. Ottaviani gave the classification (in arbitrary characteristic) of the very few Veronese embeddings of $\mathbf{P}^n$ which are tangentially $[a_n, \dots, a_1]$-defective ([1]). We will say that the numerical data $[b_n, \dots, b_1$ is a sliding of the numerical datum $[a_n, \dots, a_1]$ if $\sum_{i=1}^{j} b_i \le \sum_{i=1}^{j} a_i$ for all $1 \le i \le n-1$ and $\sum_{i=1}^{n} b_i = \sum_{i=1}^{n} a_i$. We will say that $[b_n, \dots, b_1]$ is strictly weaker than $[a_n, \dots, a_1]$ if $[b_n, \dots, b_1] \ne [a_n, \dots, a_1]$ and there is a sliding $[c_n, \dots, c_1]$ of $[a_n, \dots, a_1]$ such that $b_i \le c_i$ for all i. We will say that X strongly minimal tangentially $[a_n, \dots, a_1]$-defective if it is tangentially $[a_n, \dots, a_1]$-defective, but is not tangentially $[b_n, \dots, b_1]$-defective for any numerical datum $[b_n, \dots, b_1]$ strictly weaker than $[a_n, \dots, a_1]$. First we will use [2] to handle the case $\dim(X) = 3$.

Theorem 1. *Assume char$(\mathbb{K}) = 0$. Fix an integer $k \ge 2$ and a numerical datum $[a_3, a_2, a_1]$ for the dimension 3 such that $a_1 + a_2 + a_3 = k + 1$. $X \subset \mathbf{P}^r$ is strongly minimal tangentially $[a_3, a_2, a_1]$-defective if and only if either $a_2 = a_1 = 0$ and the triple (X, r, k) is in one of the 14 cases listed in* [2]*, Th. 0.1.*

Theorem 2. *Arbitrary char$(\mathbb{K})$. Assume the existence of an integer $t \ge 2$ such that X is tangentially $[t, 0, \dots, 0]$-defective and call t the minimal such integer. Let m be the maximal integer such that $m \ge 2$ and X is tangentially $[m, 0, \dots, 0]$-defective, i.e. the minimal positive integer such that $\langle Z \rangle = \mathbf{P}^r$ for a general scheme $Z \subset X$ associated to the numerical datum $[m, 0, \dots, 0]$. Such an integer exists because X is non-degenerate. For all integers i such that $1 \le i \le m$ set $\delta(X, i) := h^1(\mathbf{P}^r, \mathcal{I}_{Z_i}(1))$ for a general scheme $Z_i \subset X$ associated to the numerical datum $[i, 0, \dots, 0]$. Hence $\delta(X, 1) = 0$ and $\delta(X, i) > 0$ for all $2 \le i \le m$. Fix an integer i such that $b \le i \le m-1$. For all integers j such that $1 \le j \le n-1$ let $y[i,j]$ be the numerical datum $[a_{i,j,n}, \dots, a_{i,j,1}]$ with $a_{i,j,n} := i$, $a_{i,j,h} := 0$ for $h \notin \{n, j\}$ and $a_{n,j,j} := 1$. Let $Z_{i,j} \subset X$ be the general subscheme associated to the numerical datum $y[i,j]$. Then $h^1(\mathbf{P}^r, \mathcal{I}_{Z_{i,j}}(1)) = \delta(X, i) + \max\{0, \delta(X, i+1) - \delta(X, i) + j - n\}$ for all $2 \le i \le m-1$. Set $\tau := r + 1 - h^0(\mathbf{P}^r, \mathcal{I}_{Z_m}(1))$. Then $h^1(\mathbf{P}^r, \mathcal{I}_{Z_{m,j}}(1)) = \delta(X, m)$ for $1 \le j \le \tau$ and $\langle Z_{m,j} \rangle = \mathbf{P}^r$ if $j \ge \tau$.*

Remark 1. Assume char$(\mathbb{K}) = 0$. This assumption implies that a general curvilinear zero-dimensional scheme with fixed length $c > 0$ on any integral projective variety Y imposes $\min\{c, \dim(W) + 1\}$ independent conditions to any fixed linear system W on Y ([3]). A tangent vector to a smooth point of X is curvilinear. Hence if X is tangentially $[a_n, \dots, a_1]$-defective, then it is tangentially $[a'_n, \dots, a'_1]$-defective, where $a'_i := a_i$ if $i \ge 2$ and $a'_1 = 0$. Hence if X is minimally tangentially $[a_n, \dots, a_1]$-defective, then $a_1 = 0$.

Lemma 1. *Arbitrary char*$(\mathbb{K})$. *Fix* $a_i \geq 0$, $1 \leq i \leq n$, *such that* $a_1 + \cdots + a_n > 0$ *and that* X *is minimally tangentially* $[a_n, \dots, a_1]$*-defective. Assume the existence of an integer* j *such that* $1 \leq j \leq n-1$, *and* $a_j > 0$. *Set* $b_i := a_i$ *if either* $i \geq j+2$ *or* $1 \leq i < j$, $b_{j+1} := a_{j+1}+1$ *and* $b_j := a_j - 1$. *Set* $c_n := a_n + 1$, $c_i = a_i$ *if* $i \notin \{n, j\}$ *and* $c_j := a_j - 1$. *Let* $W \subset X$ *(resp* Z*, resp* E*) be a general subscheme associated to the numerical datum* $[b_n, \dots, b_1]$ *(resp.* $[a_n, \dots, a_1]$*, resp.* $[c_n, \dots, c_1]$*). Then* X *is tangentially* $[b_n, \dots, b_1]$*-defective and* $[c_n, \dots, c_1]$*-defective,* $h^1(\mathbf{P}^N, \mathcal{I}_W(1)) = h^1(\mathbf{P}^N, \mathcal{I}_Z(1) + 1 = h^1(\mathbf{P}^N, \mathcal{I}_M(1)) - j - 1 + n$ *and* $h^0(\mathbf{P}^N, \mathcal{I}_W(1)) = h^0(\mathbf{P}^N, \mathcal{I}_Z(1)) = h^0(\mathbf{P}^N, \mathcal{I}_M(1))$.

Proof. Since length(W) = length$(Z) + 1$ = length$(E) - n + j$, it is sufficient to prove that $h^0(\mathbf{P}^N, \mathcal{I}_W(1)) = h^0(\mathbf{P}^N, \mathcal{I}_Z(1)) = h^0(\mathbf{P}^N, \mathcal{I}_E(1))$. Since $Z \subset W \subset E$, it is suffficient to prove that $h^0(\mathbf{P}^N, \mathcal{I}_Z(1)) = h^0(\mathbf{P}^N, \mathcal{I}_E(1))$. Fix general $P_{i,h} \in X$, $1 \leq h \leq n$, $1 \leq j \leq a_i$. Let M be the unions of $a_1 + \cdots + a_n - 1$ closed zero-dimensional subschemes such that $M_{red} = \{P_{i,j}\}_{1 \leq i \leq n, 1 \leq h \leq a_i, (i,j) \neq (j, a_j)}$ and the connected component $M_{i,h}$ of M supported by $P_{i,h}$ is a general element of $\Lambda(P_{i,h}, i)$. The minimaly property for $[a_n, \dots, a_1]$ gives $h^1(\mathbf{P}^N, \mathcal{I}_M(1)) = 0$. For any integer x such that $0 \leq x \leq n+1$ take a general $A_x \in \Lambda(P_{j,a_j}, x)$. We only need that a single A_x is general and so we choose such schemes A_x's so that $A_x \subset A_y$ for all $x < y$. Set $E := M \cup A_{n+1}$, $W := M \cup A_{j+1}$ and $Z := M \cup A_j$. E (resp. W, resp. Z) is a general subscheme associated to the numerical datum $[c_n, \dots, c_1]$ (resp. $[b_n, \dots, b_1]$, resp. $[a_n, \dots, a_1]$). Since X is tangentially $[a_n, \dots, a_1]$-defective, $h^1(\mathbf{P}^N, \mathcal{I}_Z(1)) > 0$ and $h^0(\mathbf{P}^N, \mathcal{I}_Z(1)) > 0$. Since $Z \subset W$, we have $h^1(\mathbf{P}^N, \mathcal{I}_W(1)) \geq h^1(\mathbf{P}^N, \mathcal{I}_Z(1))$. Let t be the maximal integer x such that $0 \leq x \leq n$ and $h^1(\mathbf{P}^N, \mathcal{I}_{M \cup D_x}(1)) = 0$. Since we may choose P_{j,a_j} general in X after fixing M, we see that $t > 0$. Since X is tangentially $[a_n, \dots, a_1]$-defective, we have $t \leq a_j - 1$. Hence $h^1(\mathbf{P}^N, \mathcal{I}_{M \cup A_t}(1)) = 0$ and $h^1(\mathbf{P}^N, \mathcal{I}_{M \cup A_t}(1)) > 0$. Since lenght$(A_{t+1})$ = lenght$(A_t) + 1$, we get $h^1(\mathbf{P}^N, \mathcal{I}_{M \cup A_{t+1}}(1)) = 1$ and $h^0(\mathbf{P}^N, \mathcal{I}_{M \cup A_{t+1}}(1)) = h^0(\mathbf{P}^N, \mathcal{I}_{M \cup A_t}(1))$. Varying A_{t+1} among the elements of $\Lambda(P_{j,a_j}, t+1)$ we cover the scheme $2P_{j,a_j}$. Hence we get $h^0(\mathbf{P}^N, \mathcal{I}_{M \cup A_t}(1)) = h^0(\mathbf{P}^N, \mathcal{I}_{M \cup A_{n+1}}(1))$. □

Remark 2. Assume char$(\mathbb{K}) = 0$ and $n = 2$. If X is minimally tangentially $[a_2, a_1]$-defective, then $a_1 = 0$ (Remark 1). Hence is this case there is a complete classical classification (the $(a_n - 1)$-defective surfaces), which is explained in the introduction of [2].

Proof of Theorem 1. By Remark 1 we have $a_1 = 0$. All cases with $a_2 = 0$ are listed in [2], Th. 0.1. Assume $a_2 > 0$. Lemma 1 implies that (X, k, r) is as in one of the 14 cases of [2], Th. 0.1, with k-defectivity $\delta_k(X) \geq 2$, i.e. we only need to analyze case (2) of [2], Th. 0.1, described in detail in [2], Example 4.3 (2). As shown there, even this case has $\delta_k(X) = 1$. □

Proof of Theorem 2. Use the proof of Lemma 1. □

Remark 3. Let $X \subset \mathbf{P}^r$, $r := \binom{n+2}{2} - 1$, be the 2-Veronese embedding of $\mathbf{P}^n$. Just using that the singular locus of a quadric hypersurface, it is easy to use either the statement of Theorem 2 or the proof of Lemma 1 and the notion of sliding to get another proof of [1], Th. 1.2 and 3.2.

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