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Coherent Systems on Singular Genus One Curves

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Abstract. Here we prove the existence of α-stable coherent systems on a singular genus 1 curve. This paper is a continuation of [2].

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1. Introduction

A coherent system on a curve C is a pair (E, V) where E is a torsion free sheaf and a subspace $V \subset H^0(E)$. A coherent subsystem is a pair (F, W) where F is a subsheaf of E and $W \subset V \cap H^0(F)$. Fix a real number α and call $\mu_\alpha(E, V) = \frac{\deg(E)+\alpha \dim(V)}{\mathrm{rk}(E)}$ the α-slope of (E, V). Then a coherent system (E, V) is called stable (resp. semistable) if $\mu_\alpha(E, V) < \mu_\alpha(F, W)$ (resp. $\mu_\alpha(E, V) \leq \mu_\alpha(F, W)$) for every coherent subsystem (F, W). With this definition the moduli spaces of stable coherent systems were built (see [10]). For the background theory of coherent systems see [10] and [5]. The case of smooth genus 1 curves (i.e. elliptic curves) is investigate in [11]. In this paper we will show the existence of α-stable coherent systems on a singular genus 1 curve. This work is a continuation of a previous paper [2].

In section 2 we give the basic proprieties of torsion free sheaves and we study the elementary transformation of a polystable torsion free sheaf with pairwise non-isomorphic factors.

In section 3 we investigate the evaluation map of a polystable torsion free sheaf and apply the results to give examples of α-stable coherent systems when $\dim(V) < \mathrm{rk}(E)$.

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In section 4 we conclude giving some examples of α-stable coherent systems in other cases, namely the cases $\dim(V) = \mathrm{rk}(E)$, $\dim(V) = \mathrm{rk}(E) + 1$ and $\dim(V) = \mathrm{rk}(E) + 2$.

2. Basic proprieties of torsion free sheaves

Let C be an integral projective curve such that $p_a(C) = 1$. Assume C singular and call Q the unique singular point of C. Hence C has either an ordinary cusp or an ordinary node, i.e. either an A_2-singularity or an A_1-singularity. Let m_Q denote the maximal ideal of the local ring $\mathcal{O}_{C,Q}$.

Notation 1. Fix an integer $n \geq 1$, a rank n torsion free $\mathcal{O}_{C,Q}$-module M and a torsion free coherent sheaf F on C. There is a unique integer a such that $0 \leq a \leq n$ and $M \cong \mathcal{O}_{C,Q}^{\oplus(n-b)} \oplus m_Q^{\oplus b}$ ([7] or [6], Remark 4 at p.25). We will say that M has type b. We will say that F has local type b if the germ of F at Q has type b.

Remark 1. Fix integers r, d such that $r > 0$ and $(r, d) = 1$. There is a unique, up to isomorphism, non-locally free but torsion free sheaf $F_{r,d}$ on C with rank r and degree d. This was proved in [8] in the rank 1 case, in [12, Corollary 4.6] for any rank on a nodal curve and in [4, Theorem 22] on a cuspidal curve. We will give a different and simpler proof in Theorem 1.

Remark 2. Notice that E and E^* have the same type.

Lemma 1. *Let E be a torsion free semistable sheaf on C. Then E^* is semistable.*

Proof. Assume that E^* is not semistable and take a destabilizing sequence

$$0 \to A \to E^* \to B \to 0 \tag{1}$$

with A, B torsion free and $\mu(A) > \mu(B)$. Since C is Gorenstein, the natural map $E \to E^{**}$ is an isomorphism. For any torsion free sheaf F on C we have $Ext^i(F, \omega_C) = 0$ for every integer $i \geq 1$ ([6], Lemma 2.5.3). Hence dualizing (1) we get an exact sequence

$$0 \to B^* \to E \to A^* \to 0 \tag{2}$$

We get a contradiction to the semistability of C, because $\deg(F^*) = -\deg(F)$ for any torsion free sheaf on C ([6], Prop. 3.1.6, part 2); here we use that C is Gorenstein. □

Lemma 2. *Fix $P \in C_{reg}$ and integers $n \geq 2$, $n-1 \leq a \leq n$ and $0 < d < n$. Take a pairwise non-isomorphic $L_i \in Pic^0(C)$, $1 \leq i \leq a$, and (if $a \neq n$) set $L_n := F_{1,0}$. Set $F := \oplus_{i=1}^n L_i$. Let E be a sheaf obtained from F making d general positive elementary transformations supported by P. Then E is a semistable torsion free sheaf with degree d, rank n and type $n - a$.*

Proof. E has type $n-a$, rank n and degree d, because $P \in C_{reg}$. Assume that the statement of the lemma is not true and take an exact sequence on C:

$$0 \to A \to E \to B \to 0 \tag{3}$$

with $\mu(A) > \mu(E)$ and B torsion free. Set $r := \mathrm{rank}(A)$. Hence $0 < r < n$. Up to a twist by a degree 0 line bundle we may assume that no direct factor of F is trivial. This assumption implies $h^0(C,F) = 0$. Hence $h^1(C,F) = 0$ (Riemann-Roch). Since F is a subsheaf of E and E/F is torsion, we get $h^1(C,E) = 0$, i.e. $h^0(C,E) = d$ (Riemann-Roch). Seen F as a subsheaf of E we easily see that $\mathrm{rank}(A \cap F) = \mathrm{rank}(A)$ and $\deg(A) - d \le \deg(F \cap A) \le \deg(A)$. First assume $d = 1$. Since $h^0(C,E) = 1$ and $\deg(A) > 0$, we get $\deg(A) = 1$ and $\deg(B) = 0$. The generically surjective map $F \to B$ and the polystability of F implies that B is a direct factor of F. Since the indecomposable factors are pairwise non-isomorphic, F has only finitely many direct factors: choose a non-empty finite subset S of the set $\{1,\dots,n\}$ and take $\oplus_{i\in S} L_i$. Since E is obtained from F making a general positive elementary transformation supported by P, each $\oplus_{i\in S} L_i$ is saturated in E if $\sharp(S) < n$, contradiction. The case $d = 1$ conclude the case $n = 2$. Hence we may assume $n \ge 3$, $d \ge 2$ and use induction on n. For the fixed rank n we use induction on the degree d, i.e. we assume that the result is true for all pairs (n',d') such that either $1 \le d' < n' < n$ or $n' = n$ and $1 \le d' \le d-1$. Set $M := \oplus_{i=2}^{n} L_i$ and let N be a sheaf obtained from M making $d-1$ general positive elementary transformations supported by P. Since $d-1 < n-1$, the inductive assumption implies the semistability of N. By the openness of semistability it is sufficient to show the existence of a semistable sheaf obtained from F making d positive elementary transformations supported by P. Hence it is sufficient to prove the existence of a semistable sheaf G obtained from $L_1 \oplus N$ making a positive elementary transformation supported by P. Take any sheaf G obtained from $L_1 \oplus N$ making a positive elementary transformation supported by P. G fits in an exact sequence

$$0 \to N \to G \to L_1(P) \to 0 \tag{4}$$

Since $a > 0$, $L_1(P)$ is a degree 1 line bundle. Take any exact sequence (4). There is a sheaf G' obtained from G making a negative elementary transformation supported by P and fitting in an exact sequence

$$0 \to N \to G' \to L_1 \to 0 \tag{5}$$

Since L_1 is locally free, $\mathit{Ext}^1(L_1,N) = 0$. Since L_1 is locally free and N is semistable of degree $d-1 > 0$, $\mathit{Hom}(L_1,N)$ is semistable and of degree $d-1 > 0$. The duality for locally Cohen-Macaulay schemes and the triviality of ω_C gives $h^1(C,\mathit{Hom}(L_1,N)) = h^0(C,\mathit{Hom}(L_1,N)^*)$. Lemma 1 implies the semistability of $\mathit{Hom}(L_1,N)^*$. Hence $h^0(C,\mathit{Hom}(L_1,N)^*) = 0$. The local-to-global spectral sequence for the global Ext^1 functor gives $\mathrm{Ext}^1(L_1,N) = 0$. Thus the extension (5) splits. We just checked that every sheaf G fitting in (4) may be obtained from $L_1 \oplus N$ making a positive elementary transformation supported by P. Take a general G fitting in an exact sequence (4) and assume that G is not semistable. Fix an exact sequence on C:

$$0 \to D \to G \to M \to 0 \tag{6}$$

with $\mu(D) > d/n$, M torsion free and D with maximal slope. Hence D is semistable. Set $s := \mathrm{rank}(D)$ and $D' := L_1 \oplus D \cap M$. Let $\pi : G \to L_1(P)$ the surjective map in (4). First assume $D \subseteq N$. Since N is semistable, $\mu(D) \le \mu(N)$. Since $\mu(G) > \mu(N)$, we get a contradiction. Thus $\pi(D) \neq 0$. First assume $\pi(D) \neq L_1(P)$, i.e. $\deg(\pi(D)) \le 0$. If $s = 1$, we have $D \cong \pi(D)$ and hence $\deg(D) \le 0$, contradiction. Hence $s \ge 2$ and the sheaf $D \cap N$ has rank $s-1$. Since N is semistable, we get $\deg(D) = \deg(D \cap N) + \deg(\pi(D)) \le (s-1)(d-1)/(n-1) < sd/n$ (since $d < n$), contradiction. Now assume $\pi(D) = L_1(P)$. If $s = 1$ this implies that (5) splits, contradicting the existence of a non-trivial extension (5), because $\mu(N) < \mu(L_1(P))$. Hence we may assume $s \ge 2$. We have an exact sequence

$$0 \to N \cap D \to D \to L_1(P) \to 0 \tag{7}$$

with $\mu_+(N \cap D) \le (d-1)/(n-1)$. The proof just given (which uses the local freeness of $L_1(P)$) and the semistability of N gives $D \cong (D \cap N) \oplus L_1(P)$. Since $D \subset G$, again we get that (5) splits, contradiction. □

Corollary 1. *Fix integers n, d, a such that $n > 0$ and $n-1 \le a \le n$. If $a = n$ let F be a general element of $Pic^0(Y)^{\oplus n}$. If $a = n-1$ let F be the direct sum of $F_{1,0}$ and a general element of $Pic^0(Y)^{\oplus(n-1)}$. There is a polystable torsion free sheaf E on C such that rank$(E) = n$, $\deg(E) = d$, E has type $(a, n-a)$, the indecomposable factors of E are pairwise non-isomorphic and E is obtained from F making d positive elementary transformations..*

Proof. Twisting with a line bundle we reduce to the case $0 \le d \le n-1$. If $d = 0$, then use a direct sum of a pairwise non-isomorphic line bundles of degree 0 and (if $a \neq n$) the unique rank 1 and degree 0 torsion free sheaves. There are infinitely many such sheaves, because $\dim(\mathrm{Pic}^0(C)) = 1$ and we may twist (if necessary) all the locally free factors by general $(L_1, \dots, L_n) \in \mathrm{Pic}^0(C)^{\oplus n}$. If $(n, d) = 1$, then use Lemma 2 and that if $(n, d) = 1$ stability and semistabily are the same. Now assume $(n, d) > 1$. Set $n' := n/(n, d)$ and $d' := d/(n, d)$. Fix integers a_i, $1 \le i \le (n, d)$, such that $n' \le a_i \le n'$ and $\sum_{i=1}^{(n,d)} a_i = a$. Hence $a_i = n'$, except at most for one index i. Since $(n', d') = 1$, we just proved the existence of stable torsion free sheaves F_i such that F_i has degree d' and type $a_i, n' - a_i$. Take a general $(L_1, \dots, L_{(n,d)}) \in \mathrm{Pic}^0(C)^{\oplus(n,d)}$ and set $E := \oplus_{i=1}^{(n,d)} F_i \otimes L_i$. □

Theorem 1. *Fix integers r, d such that such that $r > 0$ $(r, d) = 1$. Let L be a line bundles of degree d. Then:*

(i) *there exists a unique stable vector bundle of rank r and determinant L;*

(ii) *there exists a unique stable torsion free sheaf of rank r, degree d; it has type* 1*;*

(iii) *if $r \geq 2$ every stable torsion free sheaf E with degree d and rank r can be written in a sequence*

$$0 \longrightarrow E_1 \longrightarrow E \longrightarrow E_2 \longrightarrow 0 \tag{8}$$

where E_1 is a stable locally free sheaf and E_2 is a stable torsion free sheaf such that rank$(E_1)\deg(E) -$ rank$(E)\deg(E_1) = 1$.

Proof. The locally free case is [14], Th. 5.1. We extend his proof to the not locally free case. Let r_1, d_1 the the unique pair of integers with $r_1d - rd_1 = 1$ and $0 < r_1 < r$. Set $d_2 := d - d_1$ and $r_2 := r - r_1$. Fix a stable vector bundle E_1 of degree d_1 and rank r_1. Let E_2 be the unique stable torsion free sheaf of degree d_2 and rank r_2 which is not locally free. Since E_1 is locally free we have dim $(\mathrm{Ext}^1(E_2, E_1) = 1$. Consider a non trivial extension

$$0 \to E_1 \to E \to E_2 \to 0.$$

As the locally free case ([14], Th. 5.1) we see that E is stable and $\dim(\mathrm{Hom}(E_1, E)) = 1$. It is clear that E is not locally free and unique. □

Notation 2. For all integers $n > 0$ and d let $U(n, d)$ denote the set of all polystable vector bundles E on C such that rank$(E) = n$, $\deg(E) = d$ and the indecomposable factors E are pairwise non-isomorphic. Let $V(n, d)$ denote the set of all polystable torsion free sheaves E on C such that rank$(E) = n$, $\deg(E) = d$, the indecomposable factors E are pairwise non-isomorphic and E is not locally free. Fix an integer $k > 0$. For all $\alpha \in \mathbb{R}$, $\alpha > 0$, let $U(\alpha; n, d, k)$ (resp. $V(\alpha; n, d, k)$) denote the set of all α-stable coherent systems (E, V) such that $E \in U(n, d)$ (resp. $E \in V(n, d)$).

Remark 3. Fix integers $n > 0$, $k > 0$ and d. $U(n, d)$ is a non-empty irreducible variety of dimension (n, d). $V(n, d)$ is a non-empty irreducible variety of dimension $(n, d) - 1$. Notice that each $F \in V(n, d)$ has type 1. The main difference with respect to the locally free case is that for any stable torsion free sheaf G non locally free we have $G \otimes L \cong G$ for all $L \in \mathrm{Pic}(C)$. Now assume $d \geq k$. Notice that $h^0(C, E) = d$ for all $E \in U(n, d) \cup V(n, d)$. Assume $U(\alpha; n, d, k) \neq \emptyset$ (resp. $V(\alpha; n, d, k) \neq \emptyset$) and take $(E, V) \in U(\alpha; n, d, k)$ (resp. $(E, V) \in V(\alpha; n, d, k)$). Since α-stability is an open condition, (E, W) is α-stable for a general k-dimensional linear subspace W of $H^0(C, E)$. Similarly, the same is true for a general $E' \in U(n, d)$ (resp. $E' \in V(n, d)$). Hence if $U(\alpha; n, d, k) \neq \emptyset$ (resp. $V(\alpha; n, d, k) \neq \emptyset$), then $U(\alpha; n, d, k)$ (resp. $V(\alpha; n, d, k)$) is irreducible and of dimension $(n, d)+k(d-k)$ (resp. $(n, d) - 1 + k(d - k)$).

Remark 4. Let Y be an integral projective curve. Set $g := p_a(Y)$. Let E be a vector bundle. If $g \geq 2$, then E is a limit of a flat family of stable vector bundles ([3], Lemma 2.4, and [9], Prop. 2.1 and Cor. 2.2). If $g = 1$, then the proof of [9], Prop. 2.1 and Cor. 2.2, gives that E is a limit of a flat family of semistable vector bundles; indeed, it is sufficient to use that semistability is an open condition.

Lemma 3. *Fix integers* r, a *such that* $r > 0$ *and any* $F \in V(r, a)$*. Let* $\mathbb{K}_Q$ *denote the skyscraper sheaf on* C *supported by* Q *and with* $h^0(C, \mathbb{K}_Q) = 1$*. Let* $u : F \to \mathbb{K}_Q$ *be a general surjection. Then* $E := Ker(u)$ *is a semistable vector bundle.*

Proof. Since F has local type 1, either E is locally free or it has local type 2. Since $\mathbb{K}_Q$ has finite support, the restriction map $H^0(C, Hom(F, \mathbb{K}_Q)) \to \mathrm{Hom}(F|\{Q\}, \mathbb{K})$ is bijective. Hence E is locally free for general u. Notice that $H^0(C, Hom(F, \mathbb{K}_Q)) \cong \oplus_i H^0(C, Hom(F_i, \mathbb{K}_Q))$, where $\oplus_i F_i$ is a decomposition of F into irreducible factors. Since E has only finitely many direct factors, the generality of u shows that no F_i is a factor of E. Hence each indecomposable factor of E has slope $< a/r$. Hence $h^0(C, Hom(E, F)) = r$. Notice that $h^0(C, Hom(M, F)) = r$ for every semistable vector bundle on C with degree $a - 1$ and rank r. Take a flat family $\{E_t\}_{t \in T}$ of vector bundles on C parametrized by an integral curve T such that there is $o \in T$ such that $E_o \cong E$ and E_t is semistable for all $t \in T\backslash\{o\}$ (Remark 4). We saw that the integer $h^0(C, Hom(E_t, F))$, $t \in T$, does not depend from $t \in T$. Hence these Hom-cohomology groups fit together to form a rank r vector bundle on T. In particular their total space is irreducible. Since there is $E_o \to F$ which is injective, there an inclusion $j_t : E_t \to F$ for general t. Since E_t is locally free, while F is not locally free, E_t and F have the same rank and $\deg(E_t) = \deg(F) - 1$, the injectivity of j_t implies $\mathrm{Coker}(j_t) \cong \mathbb{K}_Q$. Since u is general and semistability is an open condition, E is semistable. □

Lemma 4. *Fix any* $E \in U(r, d)$ *and let* G *be a general extension of* $\mathbb{K}_Q$ *by* E*. Then* G *is torsion free of type* 1 *and semistable.*

Proof. A general extension of $\mathbb{K}_Q$ by E is torsion free. Any torsion free extension of $\mathbb{K}_Q$ by E has type 1. Since C is Gorenstein, any torsion free sheaf A on C is reflexive, i.e. the natural map $A \to A^{**}$ is an isomorphism. Hence $G \cong G^{**}$. By Lemma 1 it is sufficient to prove that G^* is semistable. By Lemma 1 it is sufficient to prove that G^* is semistable. Fix a general extension

$$0 \to E \to G \to \mathbb{K}_Q \to 0 \tag{9}$$

Since G^* is torsion free, dualizing (9) we get an exact sequence

$$0 \to G^* \to E^* \to \mathbb{K}_Q \to 0 \tag{10}$$

Consider the general surjection $u : E^* \to \mathbb{K}_Q$. Lemma 3 gives the semistability of the vector bundle $\mathrm{Ker}(u)$ is semistable. Hence to prove the lemma it is sufficient to note that any surjection u gives an exact sequence (10) (for some G^* of type 1) and that dualizing it we get an exact sequence (9) (again, because the torsion freeness of $E = E^{**}$ implies that the associated map $E^{**} \to G^{**}$ is injective). □

Lemma 5. *Fix integers r, d such that $r > 0$. Then there exist $E \in U(r,d)$, $F \in U(r,d+1)$ and an injective map $j : E \to F$. We may take as E (resp. F) a general element of $U(r,d)$ (resp. $U(r,d+1)$).*

Proof. Write $d = cr + e$ with $c, e \in \mathbb{Z}$ and $0 \le e \le r-1$. If $e = 0$, then this is Corollary 1 for one positive elementary transformation and $a = n$ (i.e. taking $M := \oplus_{i=1}^{r} L_i$ with $L_i \in \mathrm{Pic}^c(C)$, $L_i \ncong L_j$ for all $i \neq j$, as E. If $0 < e \le r-2$, then E is obtained from M making e general positive elementary transformation, while G is obtained from E making a further general positive elementary transformation. Now assume $e = r-1$. Set $G := \oplus_{i=1}^{r} R_i$ with $R_i \in \mathrm{Pic}^{c+1}(C)$, $R_i \ncong R_j$ for all $i \neq j$. E is obtained from G making a general negative elementary transformation. Apply the dual of Lemma 2 or of Corollary 1 to get that $E \in U(r,d)$. The last statement follows from the proof and the openness of semistability. □

Lemma 6. *Fix integers r, d such that $r > 0$. Then there exist $E \in V(r,d)$, $F \in V(r,d+1)$ and an injective map $j : E \to F$. We may take as E a general element of $V(r,d)$.*

Proof. Copy the proof of Lemma 5, just quoting the case $a = n-1$ instead of the case $a = n$ of Lemma 2 and Corollary 1. The last statement follows from the proof and the openness of semistability. □

Iterating a times Lemmas 5 and 6 we get the following results.

Lemma 7. *Fix integers r, d, a such that $r > 0$ and $a > 0$. Then there exist $E \in U(r,d)$, $F \in U(r,d+a)$ and an injective map $j : E \to F$. We may take as E (resp. F) a general element of $U(r,d)$ (resp. $U(r,d+a)$).*

Lemma 8. *Fix integers r, d, a such that $r > 0$ and $a > 0$. Then there exist $E \in V(r,d)$, $F \in V(r,d+a)$ and an injective map $j : E \to F$. We may take as E (resp. F) a general element of $V(r,d)$ (resp. $V(r,d+a)$).*

Proposition 1. *Fix integers r, d such that $r > 0$. Then there exist $E \in U(r,d)$, $G \in V(r,d+1)$ and an injective map $j : E \to G$. We may take as E a general element of $U(r,d)$ or as G a general element of $V(r,d+1)$.*

Proof. Take any triple (A, B, m) with A locally free, B of type 1, $u : A \to B$ injective, $\mathrm{rank}(A) = \mathrm{rank}(B)$ and $\deg(B) = \deg(A) - 1$. Any such triple (A, B, m) induces a non-split exact sequence

$$0 \to A \to B \to \mathbb{K}_Q \to 0 \tag{11}$$

Conversely, any triple (A, B, m) as above gives a non-split exact sequence (11). Fix $P \in C_{reg}$ and take any sheaf B' obtained from B making a positive elementary transformation support by P. Hence there is an inclusion $m' : B \to B'$. The saturation A' of $m' \circ m(A)$ in B' is a vector bundle obtained from A making a positive elementary transformation supported by P. By construction there in an injection $A' \to B'$. Now we want to show that every positive elementary transformation τ supported by P arises in the previous way. τ from a negative elementary transformation of A^*, i.e. from a surjection $u : A^* \to \mathbb{K}_P$, just dualizing $\mathrm{Ker}(u)$. Since $P \neq Q$, the inclusion m induces an isomorphism $B^*|\{P\} \to A^*|\{P\}$. Thus u induces a negative elementary transformation of B^* supported by P, i.e. a positive elementary transformation supported by P. Write $d = cr + e$ with $a, e \in \mathbb{Z}$ and $0 \leq e \leq r-1$. Since $(r, 1) = 0$, the case $e = 0$ is true by Lemma 4. We fix a solution $E' \in U(r, cr)$ and $G' \in V(ar, 1)$. Notice that E' is a direct sum of r pair-wise non-isomorphic line bundles with the same degree, c. Now assume $e > 0$. Let E be a general vector bundle obtained from E' making e general positive elementary transformations. Corollary 1 gives $E \in V(r, d)$. Let G be the torsion free sheaf of type 1 obtained from G' making the e positive elementary transformations induced (as in the first part of the proof) from the e positive elementary transformations made to obtain E from E'. In the first part of the proof we saw that the inclusion $E' \to G'$ induces an inclusion $j : E \to G$. Since G' is a general extension of $\mathbb{K}_Q$ by E', G is a general extension of $\mathbb{K}_Q$ by E. Lemma 4 gives the semistability of G. To conclude the proof it is sufficient to prove that a general sheaf obtained from E' making first an extension by $\mathbb{K}_Q$ and the e positive elementary transformations is an element of $V(r, d+1)$. we know that it is semistable. Hence we may assume $h := (r, d+1) > 0$. We use induction on r, the case $r = 1$ being obvious. By semicontinuity it is sufficient to prove the existence of $Z \in V(r, d+1)$ which is a torsion free extension of E' by $\mathbb{K}_Q$ and e extensions $\mathbb{K}_{P_i}$, $1 \leq i \leq e$, $P_i \in C_{reg}$ for all i. Write $E' := \oplus_{i=1}^{r} L_i$, $L_i \in \mathrm{Pic}^c(C)$ for all i, and $E_j := \oplus_{i=1+(j-1)h}^{jh} L_i$. Hence $E' = \oplus_{i=1}^{h} E_i$. Apply to each E_i, $1 \leq i \leq h-1$, $(e+1)/h$ general positive elementary transformation to get $G_i \in U(r/h, (ar+e+1)/h)$ (Corollary 1). Take as G_h the sheaf obtained applying $(e+1)/h-1$ general positive elementary extensions to a general extension of $\mathbb{K}_Q$ by E_h. $E_h \in V(ar/h, (ar+e+1)/h)$ by the inductive assumption. Set $Z := \oplus_{i=1}^{h} G_i$. Taking general the first $(h-1)(e+1)/h$ positive elementary transformations we may assume that $\det(G_i) \ncong \det(G_j)$ for all $1 \leq i < j \leq h-1$. Hence $Z \in V(r, d+1)$. The last statement follows from the proof and the openness of semistability. □

3. Coherent systems with $k < n$

Remark 5. Fix integers $d \geq n > 0$ and a semistable torsion free sheaf E on C with rank n and degree d. Hence $h^0(C, E) = d$ and $h^1(C, E) = 0$. Fix any integer m such that $0 < m \leq n$ and take a general m-dimensional linear subspace V of $H^0(C, E)$. It is easy to check that the evaluation map $e_{E,V}$ is injective.

Lemma 9. *Fix integer $d > n > 0$. Let E be a semistable torsion free sheaf on C with degree d and rank n. Then $H^0(C, E)$ spans E at each point of C_{reg}.*

Proof. Fix $P \in C_{reg}$. Hence $E(-P) \in V(n, d-n)$. Thus $h^1(C, E(-P)) = 0$. Thus $H^0(C, E)$ spans E at each point of C_{reg}. □

Lemma 10. *For every integer $n \geq 1$, $F_{n,n+1}$ is spanned.*

Proof. Assume that the result is not true and call $A \subsetneq F_{n,n+1}$ the image of the evaluation map. We have $h^0(X, F_{n,n+1}) = n+1$. Lemma 9 gives that $F_{n,n+1}$ is spanned outside Q. Hence $h^0(X, A) = n+1$, $\mathrm{rank}(A) = n$ and $\deg(A) \leq n$. Set $a := \deg(A)$. Since $h^0(C, A) = n+1$, Riemann-Roch and Serre duality give $h^0(C, Hom(A, \mathcal{O}_X)) = n+1-a > 0$. Since $\mathrm{rank}(A) \leq n$ and A is spanned, we get $a \geq 1$ and $A \cong \mathcal{O}_C^{\oplus(n+1-a)} \oplus B$ with $h^0(X, B) = a$, $\deg(B) = a$, $h^1(C, B) = 0$ and $\mathrm{rank}(B) = a-1$. Hence if $a \geq 2$, then $\mu(B) = a/(a-1) > \mu(F_{n,n+1})$, contradicting the stability of $F_{n,n+1}$. Now assume $a = 1$, i.e. $A \cong \mathcal{O}_C^{\oplus n}$. Since $h^0(C, A) = n+1$, we get a contradiction. □

Since every $E \in U(n, n+1)$ is stable, the same proof gives the following result.

Lemma 11. *For every integer $n \geq 1$ every $E \in U(n, n+1)$ is spanned.*

Lemma 12. *Fix integers $d > n > 0$. There are $E \in U(n,d)$ and $F \in V(n,d)$ such that E and F are spanned. For a general $(n+1)$-dimensional linear subspace $V \subseteq H^0(C, V)$ (resp. $W \subseteq H^0(C, F)$) the map evaluation map $e_{E,V}$ (resp. $e_{F,W}$ is surjective.*

Proof. Lemma 9 shows that all elements of $U(n,d) \cup V(n,d)$ are spanned at each point of C_{reg}. Now we will show the existence E is spanned at Q. E has local type 0 or 1. If $d = n+1$, then all elements of $U(n, n+1) \cup V(n, n+1)$ are spanned at Q (Lemmas 11 and 10). Assume $d \geq n+2$ and that the results is true for the integer $d' := d-1$. Take general $E' \in U(n, d-1)$ and $F' \in U(n, d-1)$. The proofs of Lemmas 5 and 6) show the existence of inclusions $j : E' \to E$, $i : F' \to F$) such that $E \in U(n,d)$, $F \in V(n,d)$, and Q is neither in the support of $\mathrm{Coker}(j)$ nor in the support of $\mathrm{Coker}(i)$. Since E' and F' are spanned at Q, the existence of the maps i, j gives that E and F are spanned at Q. Since spannedness is an open conditions in flat families of sheaf with constant cohomology, the general members of $U(n,d)$ and of $V(n,d)$ are spanned. The last part of the lemma is straightforward. □

Lemma 13. *Fix integers $d > n > 0$ and $E \in U(n,d) \cup V(n,d)$. Then E is spanned. For a general $(n+1)$-dimensional linear subspace $V \subseteq H^0(C, V)$ the map evaluation map $e_{E,V}$ is surjective.*

Proof. Lemma 9 gives that E is spanned at each point of C_{reg}. Assume that E is not spanned at Q and let $A \subsetneq E$ be the image of $e_{E,H^0(C,E)}$. Since E is spanned at each point of C_{reg}, A has rank n and the inclusion $j : A \to E$ has cokernel supported by Q. Set $a := \deg(A)$. Hence $a < d$. Since $h^0(C, A) = h^0(C, E) = d$, we have $h^1(C, E) = d-a$, i.e. $h^0(C, Hom(E, \mathcal{O}_C)) = d-a$. Since A is spanned, the last equality implies $A \cong \mathcal{O}_C^{\oplus(d-a)} \oplus B$ for some torsion free sheaf B with degree a, rank $n-d+a$ and $h^1(C, B) = 0$. Since $a > 0$, $B \neq 0$, i.e. $n-d+a > 0$. Since $0 < a < d$ and $d > n$, $a/(n-d+a) > d/n$, i.e. $\mu(B) > \mu(E)$, contradicting the semistability of E. The last part is straightforward, because E has local type 0 or 1 at Q. □

Lemma 14. *Fix integers $d \geq 2n > 0$. Let E be any semistable torsion free sheaf on X with rank n and degree d. Then E is spanned. Let a be the type of E and set $u := \max\{n+1, n+a\}$. Then for a general u-dimensional linear subspace of $H^0(C,E)$ the evaluation map $e_{E,V}$ is surjective.*

Proof. We have $h^0(C,E) = d \geq 2n \geq u$. The proof of Lemma 9 gives that E is spanned outside Q. There are infinitely many degree 2 Cartier divisors Z of C such that $Z_{red} = \{Q\}$. Fix a general such Z. Since $\deg(E(-Z)) = d - 2n \geq 0$ and $E(-Z)$ has no trivial factor even when $d = 2n$ by the generality of Z, $h^1(C, E(-Z)) = 0$. Hence the restriction map $H^0(C,E) \to H^0(C, E|Z)$ is surjective. Since Z is zero-dimensional, the restriction map $H^0(C,E|Z) \to H^0(C,E|\{Q\}$ is surjective. Hence E is spanned at Q. Since E is locally free outside u, the last assertion is a straightforward consequence of the spannedness of E and of the inequality $h^0(C,E) \geq u$. □

Now we give another proof of Lemma 13 without the assumption that the factors of E are pairwise non-isomorphic.

Proposition 2. *Fix integers $d > n > 0$. Let E be any polystable torsion free sheaf with, degree d and rank n. Then E is spanned. If E has type ≤ 1, then for a general $(n+1)$-dimensional linear subspace of $H^0(C,E)$ the evaluation map $e_{E,V}$ is surjective.*

Proof. Lemma 9 shows that E is spanned, except at Q. If $d \geq 2n$, then use Lemma 14. Assume $n < d < 2n$. Hence $n \geq 2$. We also assume that the result is true for all pairs (n', d') such that $d' > n'$ and $1 \leq n' < n$. First assume E stable. Hence $(n,d) = 1$. By part (iii) of Theorem 1 there is an exact sequence

$$0 \to E_1 \to E \to E_2 \to 0 \tag{12}$$

with E_1 and E_2 stables, E_1 locally free, $1 \leq \mu(E_1) < 2$ and $1 < \mu(E_2) < 1$. Hence $h^1(C, E_1) = 0$. Thus E is spanned if E_1 and E_2 are spanned. The inductive assumption gives that E_2 is spanned and that E_1 is spanned, unless $\mu(E_1) = 1$. Assume $\mu(E_1) = 1$, i.e. assume $\deg(E_1) = \mathrm{rank}(E_1)$. Since E_1 is stable, we get $\mathrm{rank}(E_1) = \deg(E_1) = 1$. Since E_1 is locally free, it has its only section does not vanishes at Q. Hence E_1 is spanned at Q. Since $h^1(C, E_1) = 0$ and E_2 is spanned, E is spanned at Q. Now assume E polystable, but not stable, say $E = \oplus_{i=1}^{j} F_j$ with $s \geq 2$, F_j stables and $\mu(F_j) = \mu(E)$ for all E. Hence $\mu(F_j) > 1$. Since F_j is stable and of rank $n/s < n$, F_j is spanned. Hence $E = \oplus_{i=1}^{j} F_j$ is spanned. For the last assertion use that E has type ≤ 1. □

Proposition 3. *Fix integers k, d, such that $k > 0$ and either $k < d$ or $k = d$ and $(n,d) = 1$. Fix a polystable torsion free sheaf E with pairwise non isomorphic summands of rank n and degree d. Then for a generic $V \subset H^0(E)$ the coherent system (E,V) is 0^+-stable.*

Proof. The proof is essentially the same of the smooth case ([11], Theorem 4.1). A coherent subsystem (F, W) which violates the 0^+-stability must have $\mu(F) = \mu(E)$ i.e. F must be a sum of factors of E. In particular we have a finite number of such subsheaves (if (n,d)=1 we have nothing to prove). For those subsheaves we have to check

$$\frac{\dim(H^0(F) \cap V)}{\mathrm{rank}(F)} < \frac{k}{n}. \tag{13}$$

Since V is generic we have for all subsheaves F of slope $\mu(F) = \mu(E)$

$$\frac{\dim(H^0(F) \cap V)}{\mathrm{rank}(F)} = \max\{0, h^0(F) - d + k\}$$

If this number is 0 equation (13) holds trivially. So we assume $h^0(F) = \deg F > d - k$; then equation (13) becomes

$$\frac{\deg F - d + k}{\mathrm{rank}(F)} < \frac{d - d + k}{n},$$

which holds for every proper subsheaf. □

Lemma 15. *Fix positive integers m, k and a rank m degree k polystable torsion free sheaf F on C. Let E be the general extension of F by $\mathcal{O}_C^{\oplus k}$. Then E is torsion free, polystable and with the same type as F.*

Proof. Obviously, E is torsion free and with the same type as F. We have $h^0(C, F) = k$ and for a general extension the coboundary map $\delta : H^0(C, F) \to H^1(C, \mathcal{O}_C^{\oplus k})$ is an isomorphism. As in [1] we will say that an extension of F by $\mathcal{O}_C^{\oplus k}$ us complete if the associated coboundary map δ is an isomorphism. Since $\dim(\mathrm{Ext}^1(F, \mathcal{O}_C^{\oplus k})) = k^2$ the group $GL(k) \cong \mathrm{Aut}(\mathcal{O}_C^{\oplus k})$ acts transitively on the set of all complete extension. Hence, up to isomorphisms, there is a unique such extension. $h^1(C, E) = 0$ and $h^0(C, E) = k$ for any such E. Conversely, F is uniquely determined from E, because $F \cong e_{E,H^0(C,E)}$. Hence any automorphism of E induces an automorphism of F. To check the polystability of E we will use induction on the integer $m + k$. First assume $m = 1$, while k is arbitrary. If F is locally free (resp. $F \cong F_{1,k}$), then use that for any $E \in U(k+1, k)$ (resp. $E \in V(k+1, k)$) the evaluation map $e_{E,H^0(C,E)}$is injective and it has torsion free cokernel. Now assume $m \geq 2$ and that the result is true for all integers m', k' such that $2 \leq m' + k' < m + k$. First assume $(m, k) = h > 0$. Hence $F \cong \oplus_{i=1}^h F_i$ with F_i polystable of degree k/h and rank m/h. Apply the inductive assumption to the pair of integers $(m/h, k/h)$ and take $E := \oplus_{i=1}^h E_i$ with E_i extension of F_i by $\mathbb{O}_C^{\oplus k/h}$. Now assume $(m, k) = 1$, i.e. assume that F is stable. Assume that E is not stable, i.e. (since $(m + k, k) = 1$) assume the existence of a proper saturated subsheaf G of E such that $\mu(E/G) > \mu(G)$. Thus $h^0(C, Hom(E/G, G)); 0$. Thus E is not simple and it has an endomorphism $u : E \to E$ such that $u \neq 0$, $u(E) \subseteq E_1$ and $u^2 = 0$. Since u induces an endomorphism $u_* : H^0(C, E) \to H^0(C, E)$, it induces an endomorphism $v : F \to F$. Since $u^2 = 0$ and u^2 induces v^2, $v^2 = 0$. Since F is simple, $v = 0$.

Since $v = 0$, $u(E) \subseteq \mathcal{O}_C^{\oplus k}$. However, since $h^1(C, E) = 0$, there is a non-zero morphism $E \to \mathcal{O}_C$, contradiction. □

Lemma 16. *Fix integers d, k, n such that $n > k > 0$ and $d \geq k$. Let E be a general element of $U(n, d)$ and V a general k-dimensional linear subspace of $H^0(C, E)$. Then $e_{E,V}$ is injective and Coker$(e_{E,V}) \in U(n - k, d)$.*

Proof. It is sufficient to find just one pair (E, V) for which the lemma is true. If $d = k$, use Lemma 15. Now assume $d > k$. There is $E' \in U(n, k)$ and an inclusion $j : E' \to E$ with $E' \in U(n, k)$ (Lemma 7). Furthermore, we may assume that E' is general in $U(n, k)$ (Lemma 7). Set $N := H^0(C, E')$ and use j to see M as a k-dimensional linear subspace of $H^0(C, E)$. For general E we may see j as the composition of $d - k$ general positive elementary transformations supported by general points of C_{reg}. For general (E, j), $\mathrm{Im}(e_{E,M})$ is saturated in E and $\mathrm{Coker}(e_{E,M})$ is obtained from $\mathrm{Coker}(e_{E',N})$ making $d - k$ general positive elementary transformations supported by general points of C_{reg}. Since $\mathrm{Coker}(e_{E',N})$ may be considered as a general element of $U(n - k, k)$, Lemma 7 gives the polystability of $\mathrm{Coker}(e_{E,M})$. □

Just quoting Lemma 8 instead of Lemma 7 in the proof of Lemma 16 we get the following result.

Lemma 17. *Fix integers d, k, n such that $n > k > 0$ and $d \geq k$. Let E be a general element of $V(n, d)$ and V a general k-dimensional linear subspace of $H^0(C, E)$. Then $e_{E,V}$ is injective and Coker$(e_{E,V}) \in V(n - k, d)$.*

Remark 6. Fix real numbers $\alpha > \beta$. Let (E, V) be a coherent system on C. If (E, V) is α-stable and β-stable, then it is γ-stable for all $\alpha > \gamma > \beta$. The same is true for semistablity.

Theorem 2. *Fix integers d, n, k such that $d \geq n+2$ and $n > k > 0$. Fix a general $E \in V(n, d)$ and a general k-dimensional linear subspace V of $H^0(C, E)$. Then the coherent system (E, V) is α-stable for all $0 < \alpha < d/(n - k)$.*

Proof of Theorem 2. By Remark 6 and Proposition 3 it is sufficient to prove the existence of $\epsilon > 0$ such that for every $d/(n - k) - \epsilon < \alpha < d/(n - k)$ there in a k dimensional linear subspace M of $H^0(C, E)$ such that (E, M) is α-stable. By Proposition 1 and the generality of E there is an exact sequence

$$0 \to G \xrightarrow{j} E \to \mathbb{K}_Q \to 0 \tag{14}$$

where G is a general element of $U(n, d-1)$. Let M be a general k-dimensional linear subspace of G. By [2] or Lemma 16 $\mathrm{Coker}(e_{G,M})$ is a polystable vector bundle with pairwise non-isomorphic factors. Hence we have an exact sequence

$$0 \to \mathcal{O}_C^{\oplus k} \to G \to F \to 0 \tag{15}$$

in which $F' := \mathrm{Coker}(e_{E,M}) \in V(n - k, d)$ (Lemma 17). The polystability of F' is sufficient to copy the proof of [11], Th. 4.2. □

Proposition 4. *Suppose* $0 < k < n$ *and either* $k < d$ *or* $k = d$ *and* $(n, d) = 1$*. The general* $(E, V) \in U(\alpha, n, d, k) \cup V(\alpha, n, d, k)$ *is* α*-stable for all* $\alpha \in \left(0, \frac{d}{n-k}\right)$*.*

Proof. By Lemmas 16 and 17 the coherent system is associated to a sequence

$$0 \to \mathcal{O}^k \xrightarrow{\phi} E \to G \to 0$$

where G is a polystable torsion free sheaf with non-isomorphic factors. Consider the coherent subsystem (F, W) with $\dim W = k$ and $F = \phi(V \otimes \mathcal{O})$. We obtain

$$0 \to (F, W) \to (E, V) \to (G, 0) \to 0$$

Now (F, W) and $(G, 0)$ are α-semistable coherent systems for all $\alpha > 0$ and $\mu_\alpha(F, W) = \mu_\alpha(G, 0)$ for $\alpha = \left(\frac{d}{n-k}\right)$. This proves the $\left(\frac{d}{n-k}\right)$-semistability of (E, V), so the claim follows by proposition 3 and by the fact that the set $\{\alpha \mid (E, V) \text{ is } \alpha\text{-stable}\}$ is an open interval with a finite number of critical value. □

Proposition 5. *Fix positive integers* x, k, n *such that* $n \neq k$ *and* $n \equiv 0 \pmod{k}$*. Let* E *be the direct sum of* n *rank* 1 *torsion free sheaves, all of them of degree* x*. Let* V *be a general* k*-dimensional linear subspace of* $H^0(C, E)$*. Then the coherent system* (E, V) *is* α*-stable for all* $0 < \alpha < xn/(n - k)$*.*

Proof. Write $E = \oplus_{i=1}^k E_i$ with each E_i sum of n/k of the rank 1 factors of E. Take a non-zero section of each E_i and let N be k-dimensional linear subspace spanned by these k section. The coherent system (E, M) is α-polystable for all $0 < \alpha < xn/(n - k)$. Hence if the proposition fails there is an integer m such that $k > m \geq 1$, $k \equiv 0 \pmod{m}$ and a rank n/m factor G of E such that $\dim(V \cap H^0(C, G)) = k/m$. Fix m. The set of all rank n/m factors of E is an irreducible variety T_m of dimension $n/m(n - n/m)$. The set of all k-dimensional linear subspaces of $H^0(C, E)$ is an irreducible variety Γ of dimension $k(nx - k)$. For each $G \in T_m$ let $B(G)$ be the set of all $W \in \Gamma$ such that $\dim(W \cap H^0(C, G)) \geq k/m$. $B(G)$ is an irreducible variety of dimension $k/m(xn/m - k/m) + (k - k/m)(xn - xn/m - k + k/m)$. Since $k > m$, $k(nx - k) > n/m(n - n/m) + k/m(xn/m - k/m) + (k - k/m)(xn - xn/m - k + k/m)$, concluding the proof. □

4. Some other cases

We start with the easiest case, i.e. when $\dim(V) = \mathrm{rk}(E)$.

Proposition 6. *Fix integers* $d > n > 0$ *and any* $E \in U(n, d) \cup V(n, d)$*. Let* $V \subset H^0(C, E)$ *be a general* n*-dimensional linear subspace. Then the coherent system* (E, V) *is* α*-stable for all* $\alpha > 0$*.*

Proof. The evaluation map $e_{E,V}$ is injective (Remark 5). The proof made in the smooth case ([5], Th. 5.6, or [11], Th. 5.4) works verbatim because it only uses the injectivity of $e_{E,V}$, that E is polystable and that E has only finitely many direct factors. □

Proposition 7. *Fix integers $n > 0$ and $x \geq 2$. Let E be any direct sum of n rank 1 torsion free sheaves, all of them with degree x. Let V be a general n-dimensional linear subspace of $H^0(C, E)$. Then the coherent system (E, V) is α-stable for all $\alpha > 0$.*

Proof. By Remark 6 it is sufficient to show that (E, V) is 0^+-coherent and that it is α-coherent for all $\alpha \gg 0$. it is sufficient to show that for every $\alpha > 0$ there is an n-dimensional linear subspace M of $H^0(C, E)$ such that (E, M) is α-stable. The evaluation map $e_{E,V}$ is injective (Remark 5). Write $E = \oplus_{i=1}^n L_i$ with $\mathrm{rank}(L_i) = 1$ for all i. Fix $e_i \in H^0(C, L_i)$, $e_i \neq 0$, and see e_i as a section of E. Let W be the n-dimensional linear subspace of $H^0(C, E)$ generated by $e_1, \ldots, e_n$. The coherent system (E, W) is α-polystable for all α. Hence (E, V) is α-semistable and it is not α-stable if and only if there is a proper direct factor G of E such that $\dim(V \cap H^0(C, G)) = \mathrm{rank}(G)$. Fix an integer m such that $1 \leq m \leq n-1$. E has at most $m(n-m)$ different rank m factors. Fix one of them, A. We have $h^0(X, A) = mx$. Let $G(n, H^0(C, E))$ be the Grassmannian of all n-dimensional linear subspaces of $H^0(C, E)$. $G(n, H^0(C, E))$ is an irreducible variety and $\dim(G(n, H^0(C, E))) = n(nx-n) = n^2(x-1)$. Set $B(A) := \{M \in G(n, H^0(C, E)) : \dim(M \cap H^0(C, A)) \geq m\}$. $B(A)$ is an irreducible variety of dimension $m(mx-m)+(n-m)((n-m)x-(n-m)) = m^2x+(n-m)^2x-m^2-(n-m)^2$ (a Schubert cycle). Since $m(n-m)+m^2x+(n-m)^2x-m^2-(n-m)^2 < n^2(x-1)$, a general $V \notin B(A)$ for any A and any m. □

In the next proposition we extend the technique of the smooth case the resolve the $k = n + 1$ case.

Proposition 8. *Fix integers $d \geq n + 2 > 0$ and $E \in U(n, d) \cup V(n, d)$. Let $V \subseteq H^0(C, E)$ be a general linear subspace with dimension $n + 1$. Then the coherent system (E, V) is α-stable for all $\alpha > 0$.*

Proof. Lemma 13 implies that the evaluation map $j_V : V \otimes \mathcal{O}_C \to E$ is surjective. Fix an integer r such that $0 < r < n$ and a rank r subsheaf F of E. Set $W := H^0(C, F) \cap V$. Since j_V is generically surjective and $r < n$, $W \neq V$.

Claim: The evaluation map $j_W : W \otimes \mathcal{O}_C \to E$ is injective.

Proof of the Claim: Since $W \neq V$, there is an n-dimensional linear subspace M of V such that $W \subseteq M$. It is sufficient to prove the injectivity of j_M. Assume that j_M is not injective. Then $\mathrm{Coker}(j_M)$ is a coherent sheaf of rank at least 1 which is a quotient of E. Since E is spanned by V, $\mathrm{Coker}(j_M)$ is spanned by V/M. Since $\dim(M/V) = 1 \geq \mathrm{rank}(\mathrm{Coker}(j_M))$, this implies $\mathrm{Coker}(j_M) \cong \mathcal{O}_C$. Since E is spanned, the existence of a surjective map $E \to \mathcal{O}_C$ implies that $\mathcal{O}_C$ is a direct factor of E, contradicting the semistability of E.

Since the evaluation map $j_W : W \otimes \mathcal{O}_C \to E$ is injective, F has rank r and $\mathrm{Im}(j_W) \subseteq F$, we have $\dim(W) \leq r$. Thus $\mu_\alpha(F, W) \leq \mu(F) + \alpha < \mu(E) + (1 + 1/n)\alpha = \mu_\alpha(E)$ for all $\alpha > 0$. Hence (E, V) is α-stable. □

We conclude giving examples of α-stable coherent systems in the case when $\dim(V) = \mathrm{rk}(E) + 2$.

Remark 7. Let A be a rank n polystable torsion free sheaf whose indecomposable factor are pairwise non-isomorphic. Fix an integer $m > 0$ and m distinct points $P_1, \dots, P_m \in C_{reg}$. Set $d := \deg(A)$ and $I(P_i) := H^0(C, A(-P_i))$, $1 \le i \le m$. Now we fix an integer k such that $d \ge k > n$ and a general k-dimensional linear subspace V of $H^0(C, A)$. Since V is general, $\dim(V \cap I(P_i)) = \max\{0, k-n\}$ for all i. Fix an integer r such that $2 \le r \le n$. Let $V(r)$ denote the Grassmannian of all r-dimensional linear subspace of V. Let $B(V, r)$ be the subset of $V(r)$ formed by all $M \in V(r)$ such that the evaluation map $j_M : M \otimes \mathcal{O}_C \to A$ is not injective. Set $B(V, r, P_i) := \{M \in V(r) : M \cap I(P_i) \neq \{0\}\}$. Notice that $B(V, r, P_i) := \{M \in V(r) : M \cap I(P_i) \neq \{0\}$ and $B(V, r) \subseteq \cap_{i=1}^{m} B(V, r, P_i)$.

First Claim: Assume that $(P_1, \dots, P_m, V)$ is a general $(m+1)$-ple. Then $\dim(\langle \cup_{j=1}^{i} (V \cap I(P_j)) \rangle) = \min\{k, m(k-n)\}$.

Proof of the First Claim: Since $\dim(V \cap I(P_j)) = k - n$ and $\dim(V) = k$, the inequality $\dim(\langle \cup_{j=1}^{i} (V \cap I(P_j)) \rangle) \ge \min\{k, m(k-n)\}$. Since $\dim(V \cap I(P_1)) = k - n$, the First Claim is true. Assume $2 \le i \le m$ and that the Claim is true for the integer $i' := i - 1$. Set $M := \langle \cup_{j=1}^{i-1} (V \cap I(P_j)) \rangle$. The inductive assumption gives $\dim(M) = \min\{k, (i-1)(k-n)$. Since the First Claim is true for all integers $x \ge i$ if $\dim(M) = k$, we may assume $\dim(M) = (i-1)(k-n)$. Let $N_j \subset I(P_j)$, $1 \le j \le i$, be general linear subspaces such that $\dim(N_j) = k - n$. The generality of all N_j implies $\dim(\langle \cup_{j=1}^{i-1} N_j \rangle) = (i-1)(k-n)$ and $\dim(\langle \cup_{j=1}^{i} N_j \rangle) = \min\{k, i(k-n)\}$. We fix $P_1, \dot{,} P_i$, but we change V: we take as V a general linear subspace of $H^0(C, A)$ containing each N_j, $1 \le j \le i$. For such vector space V we have $N_j = I(P_j) \cap V$ for all $1 \le j \le i$. Hence the equality $\dim(\langle \cup_{j=1}^{i} N_j \rangle) = \min\{k, i(k-n)\}$ is equivalent to the truth of the Claim for the integer i. Inductively we get the First Claim.

Now we fix integers $d \ge k > n \ge r \ge 2$ and take $m := r + 1$. Assume $k \ge (r+1)(k-n)$, i.e. assume $n < k \le (r+1)n/r$. Fix a general $(m+1)$-ple $(P_1, \dots, P_m, V)$.

Second Claim: For every $W \in V(r)$ the evaluation map is injective.

Proof of the Second Claim: Fix any $W \in V(r)$ and assume $W \in B(V, r)$. Hence $W \in \cap B(V, r, P_i)$ for all $i = 1, \dots, r+1$. Hence $W \cap I(P_i) \cap V \neq \{0\}$. The First Claim and the assumption $k \ge (r+1)(k-n)$ imply that the $r+1$ linear spaces $I(P_i) \cap V$, $1 \le i \le r+1$, are linearly independent. Hence the $r+1$ non-zero linear spaces $W \cap I(P_i) \cap V$, $1 \le i \le r+1$, cannot be contained in the r-dimensional linear space W. The contradiction proves the Second Claim.

Theorem 3. *Fix integers n, r, k, d such that $2 \le r \le n < k \le (r+1)n/r$ and $d \ge k$. Let A be a rank n polystable torsion free sheaf on C such that* $\deg(A) = d$ *and its indecomposable factors are pairwise non-isomorphic. Let V be a general k-dimensional linear subspace of $H^0(C, A)$.*

(a) *The evaluation map j_V is surjective.*

(b) *For every integer* r *such that* $2 \leq r \leq n$ *and every* r*-dimensional linear subspace* M *of* V *the evaluation map* j_M *is injective.*

(c) *Assume* $k = n + 2$, $n \geq 4$ *and either* $(d, n) = 1$ *or* $d > k$. *Then the coherent system* (A, V) *is* α*-stable for all* $\alpha > 0$.

Proof. Part (a) is Proposition 2. Part (b) is the Second Claim of Remark 7. Make all the assumptions of part (c) and fix $\alpha > 0$, an integer s such that $1 \leq s \leq n-1$ and a rank s subsheaf F of A. Set $W := V \cap H^0(C, F)$ and $r := \dim(W)$. Since $k > n$ and A is semistable, we have $\mu_\alpha(F, W) < \mu_\alpha(A, V)$ if $r \leq s$. Now assume $r \geq s+2$. Since V spans A, the $(n+2-r)$-dimensional vector space V/W spans the coherent sheaf A/F. Since A/F has generic rank $n-s$, this is possible only if $r = s+2$ and $A/F \cong \mathcal{O}_C^{\oplus(n-s)}$. Since A is spanned, the existence of a surjection $A \to \mathcal{O}_C^{\oplus(n-s)}$ implies that $\mathcal{O}_C^{\oplus(n-s)}$ is a direct factor of A, contradicting the semistability of A. Now assume $r = s+1$. We get $\mu_\alpha(A; V) > \mu(F, W)$ if $2s \geq n+1$. If $2s = n$, we need also to use that $A \in U(n, d) \cup V(n, d)$ and that $V \neq H^0(C, A)$, since $d > n$. Hence we may assume $2s < n$. Since $n \geq 4$, $r := s+1 \leq n$. The Second Claim of Remark 7 gives the injectivity of j_W. Since $W \subseteq H^0(C, F)$, this implies $s \geq r$, contradiction. □

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