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Behavior of the Trinomial Arcs $I(p,k,r,n)$ when $0 < \alpha < 1$

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Abstract

In this paper, we deal with the family $I(p,k,r,n)$ of trinomial arcs defined as the set of roots of the trinomial equation $z^n = \alpha z^k + (1-\alpha)$, with $z = \rho\, e^{i\theta}$ is a complex number, α is a real number between 0 and 1 and k is an integer such that $k = (2p+1)n/(2r+1)$, where n, p and r are three integers satisfying some conditions. These arcs $I(p,k,r,n)$ are continuous arcs inside the unit disk, expressed in polar coordinates (ρ,θ). The question is to prove that ρ changes monotonically with respect to θ and that $\rho\,(\theta)$ is a decreasing function, for each trinomial arc $I(p,k,r,n)$.

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1 Introduction

Consider the trinomial equation

$$z^n \;=\; \alpha\; z^k \;+\; (1-\alpha) \tag{1}$$

where z is a complex number, n and k are two integers such that $k = 1$, 2, ..., $n-1$ and α is a real number. Noting that the first discussion of the behavior of the roots of trinomial equation was fulfilled by Fell [4]. She has established a large description of the trajectories of these roots, called *trinomial arcs*. These arcs can be expressed in polar coordinates (ρ, θ) by a function $\rho(\theta)$ and are continuous arcs corresponding to a number α which is whether between 0 and 1, or between 1 and $+\infty$, or also between $-\infty$ and 0. In [4], Fell has studied equally the monotonicity of the function $\alpha(\theta)$ and gave one bound for the modulus of roots. However, she could not establish the monotonicity of ρ as a function of θ. In fact, the descriptive results of Fell [4] gave us the information about the form and the localization of the trinomial arcs. However, these types of arcs are not well-defined, in order to be studied. In this paper, we will restrict our attention to a family of trinomial arcs, solutions of equation (1) with $0 < \alpha < 1$, inside the unit disk $D_u = \{z\ ;\ |z| \leq 1\}$, denoted by $I(p, k, r, n)$, where p, k, r and n satisfy some conditions. By first, we formulate and define this family of trinomial curves. Notice that Dubuc and Zaoui were interested in [3] in some particular trinomial arcs denoted by B_m and which are part of this family of arcs $I(p, k, r, n)$. Next, we prove in this work that $\rho(\theta)$ is a derivable function for these arcs. With a view to solving the problem of monotonicity of $\rho(\theta)$ for the trinomial arcs $I(p, k, r, n)$, two important intermediate results are showed. At last, this study allows us to prove that $\rho(\theta)$ is a decreasing function.

2 Study of the trinomial equation

In the equation (1), fix n and k. For $z = \rho e^{i\theta}$ in (1), one has $\rho^n e^{in\theta} = \alpha\rho^k e^{ik\theta} + (1-\alpha)$. Separating real and imaginary parts, one gets $\rho^n \sin n\theta = \alpha\rho^k \sin k\theta$ and $\rho^n \cos n\theta = \alpha\rho^k \cos k\theta + (1-\alpha)$. So, when $\theta \neq l\pi/n$ where l is an integer, we get

$$\rho^{n-k} \;=\; \alpha\ \sin k\theta\ /\ \sin n\theta \tag{2}$$

On the other side, divide (1) by z^n and consider the imaginary part. When $\alpha \neq 0$ and $\theta \neq l\pi/(n-k)$ where l is an integer, we obtain that

$$\rho^k \;=\; (1 - 1/\alpha)\ \sin n\theta\ /\ \sin(n-k)\theta \tag{3}$$

Therefore, we have the next equation of the trajectories of roots of (1):

$$\rho^{n-k} \sin n\theta - \rho^n \sin(n-k)\theta = \sin k\theta \tag{4}$$

In fact, Fell has studied in [4] the trinomial equation

$$\lambda\ z^n \;+\; (1-\lambda)\ z^k \;-\; 1 \;=\; 0, \tag{5}$$

where z is a complex number, n and k are two integers such that $k = 1, 2, ..., n-1$ and λ is a real number. Substituting into equation (5) the expression given for z^n by equation (1), we get $\left(z^k - 1\right)[1 - \lambda(1-\alpha)] = 0$. So, $z^k = 1$ or $\lambda(1-\alpha) = 1$. As z is a complex number, it follows that $\alpha = 1 - 1/\lambda$. Hence, in order to pass from (1) to (5), we can set $\alpha = 1 - 1/\lambda$. From this equality stems easily that the case $0 \le \alpha \le 1$ of (1) corresponds to the case $1 \le \lambda < +\infty$ of (5).

In this work, we are interested in the case $0 \le \alpha \le 1$, we have so

$$sign(\sin n\theta) = sign(\sin k\theta) = -sign(\sin(n-k)\theta) \tag{6}$$

Definition 2.1 *An angle θ which fulfills* (6) *will be called a (n,k)-feasible angle for the trinomial equation* (1) *with $0 \le \alpha \le 1$.*

Moreover, in view of the next lemma of [2], the trajectories of roots of (1) with $0 \le \alpha \le 1$ are inside the unit disk.

Lemma 2.2 *For any (n,k)-feasible angle θ for the equation* (1) *with $0 \le \alpha \le 1$, the function of ρ, $-\rho^n\{\sin(n-k)\theta/\sin k\theta\} + \rho^{n-k}\{\sin n\theta/\sin k\theta\} - 1$, is increasing and vanishes for one and only one positive value of ρ, which is not larger than* 1*.*

Remark 2.3 *The upper and lower half-planes are symmetrical. Then, we will restrict our study of trinomial arcs to the upper half-plane.*

3 Description and definition of trinomial arcs $I(p,k,r,n)$

Notice that for $\alpha = 0$, the equation (1) has n roots; the n^{th} roots of unity. In [4], Fell tells us that the trajectories of the n roots can be described as trajectories of particles starting at these n roots. As α changes from 0 to 1, they move continuously until $\alpha = 1$, $(n-k)$ of them have moved into $(n-k)^{th}$ roots of unity and k of them have collapsed to 0. There are k trajectories going to 0, the k tangents being lines going through 0 and one k^{th} root of -1. Consider $C = \{n^{th}$ roots of unity$\}$, $D = \{(n-k)^{th}$ roots of unity$\}$ and $E = \{k^{th}$ roots of $-1\}$. Let γ be in C and δ be the unique nearest neighbor of γ in $D \cap E$. Fell ([4]) asserts that, in the case $\delta \in D \cap E$ with $0 \le \alpha \le 1$, there exists γ' in C such that δ is equidistant from γ and from γ'. There exists also α_0 in $[0,1]$ such that the trajectories of two particles starting at γ and γ' when $\alpha = 0$ are continuous arcs until the point of their meeting on the line segment $\theta = \arg(\delta)$ when $\alpha = \alpha_0$. When α moves from α_0 to 1, the two roots remain on the segment $\theta = \arg(\delta)$, one of them goes to 0 and the other tends to δ. Fell shows in [4] that all the trinomial arcs solutions of (1) in the case $0 \le \alpha \le 1$ with $\delta \in D \cap E$ are such that the feasible angles θ belong to intervals of length less than or equal to π/n and bounded on the one side by $\arg(\delta)$ with δ is both an k^{th} root of -1 and an $(n-k)^{th}$ root of unity and on the other side by $\arg(\gamma)$ with γ is an n^{th} root of unity. There are so two types of arcs in this case; the first type is such that θ belongs to $[\arg(\gamma), \arg(\delta)]$ where $\gamma \in C$ and the second type is such that θ belongs to $[\arg(\delta), \arg(\gamma')]$ where $\gamma' \in C$, such that δ is equidistant from γ and from γ'. Then, we can set $\arg(\gamma) = 2\pi r/n$ where r is a nonzero integer, it follows that $\arg(\gamma') = 2(r+1)\pi/n$. Moreover, we can put $\arg(\delta) = (2p+1)\pi/k = 2\pi q/(n-k)$ where p is an integer and q is a nonzero integer.

Lemma 3.1 *For any trinomial arc solutions of equation* (1) *with* $0 \leq \alpha \leq 1$ *and the feasible angles are bounded by* $\arg(\gamma)$ *and* $\arg(\delta)$*, where γ is an n^{th} root of unity and δ is both an k^{th} root of -1 and an $(n-k)^{th}$ root of unity, the integer k verify that $k = (2p+1)n/(2r+1) = (2p+1)n/(2[p+q]+1)$, where p is an integer and q and r are nonzero integers such that* $\arg(\delta) = (2p+1)\pi/k = 2\pi q/(n-k)$ *and* $\arg(\gamma) = 2\pi r/n$.

Proof. We assume that $\arg(\delta) = (2p+1)\pi/k = 2\pi q/(n-k)$ and $\arg(\gamma) = 2\pi r/n$, where p is an integer and q and r are nonzero integers. By first, from the equality $(2p+1)\pi/k = 2\pi q/(n-k)$ stems immediately that the integer k verify $k = (2p+1)n/(2[p+q]+1)$. In addition, according to Fell [4], there exists an n^{th} root of unity γ' such that δ is equidistant from γ and from γ'. We have so $\arg(\gamma') = 2(r+1)\pi/n$ such that $(2p+1)\pi/k - 2\pi r/n = 2(r+1)\pi/n - (2p+1)\pi/k$. Then, we deduce that the integer k satisfy $k = (2p+1)n/(2r+1)$.

Remark 3.2 *By Lemma 3.1, the integer k verify that $k = (2p+1)n/(2r+1) = (2p+1)n/(2[p+q]+1)$. Therefore, $q = r - p$. Because q is a nonzero integer, we deduce that the integers p and r satisfy the condition $r \geq p+1$.*

In [3], Dubuc and Zaoui were interested in some particular trinomial arcs denoted by B_m and defined as the set of roots of (1) with $0 \leq \alpha \leq 1$, $n = m$, $k = m-2$, where m is an odd integer larger than 2 and the feasible angles belong to the interval $[\pi - \pi/m, \pi]$. They have showed in [3] that $\rho(\theta)$ is a decreasing function on $[\pi - \pi/m, \pi]$ for the arcs B_m. Because m is an odd integer, we can say that γ such that $\arg(\gamma) = \pi - \pi/m$ is an n^{th} root of unity and δ such that $\arg(\delta) = \pi$ is both an k^{th} root of -1 and an $(n-k)^{th}$ root of unity. Dubuc and Zaoui have so solved the problem of monotonicity of $\rho(\theta)$, pointed out in [4], for some particular trinomial arcs, namely B_m, solutions of (1) in the case $0 \leq \alpha \leq 1$ with $\delta \in D \cap E$ and $\theta \in [\arg(\gamma), \arg(\delta)]$. In this paper, our objective is to study the monotonicity of $\rho(\theta)$ for all trinomial arcs corresponding to this case. So, these arcs, which will be denoted by $I(p,k,r,n)$, will be defined on the intervals of the form $[2\pi r/n,\ (2p+1)\pi/k]$ where p is an integer and r is a nonzero integer.

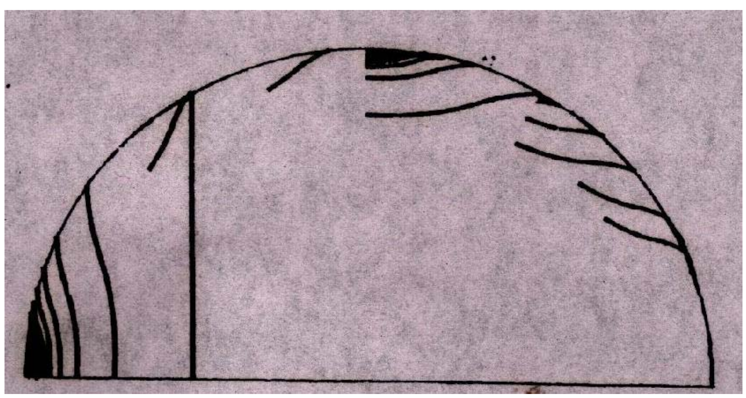
Remark 3.3 *The cases $\alpha = 0$ and $\alpha = 1$ are two particular cases for the trinomial equation* (1). *When $\alpha = 0$, equation* (1) *becomes $z^n = 1$. So, its solutions are the n^{th} roots of unity. In the case $\alpha = 1$,* (1) *becomes $z^k\left[z^{n-k} - 1\right] = 0$. Then, the n roots of* (1) *are the $(n-k)^{th}$ roots of unity, which are simple roots and* 0*; a root of multiplicity k.*

When $n = 2$, the trajectories of roots of equation (1) with $0 < \alpha < 1$ are linear, then we define the family of trinomial arcs $I(p, k, r, n)$ as follows :

Definition 3.4 *If n is an integer greater than or equal to* 3*, so $I(p, k, r, n)$ is the set of roots of equation* (1) *with $0 < \alpha < 1$ and the feasible angles belong to the interval $[2\pi r/n,\ (2p+1)\pi/k]$, where p is an integer, r is a nonzero integer verifying $r \geq p+1$ and k is an integer such that $k = (2p+1)n/(2r+1)$.*

Remark 3.5 *In the definition of $I(p, k, r, n)$, we use that $\arg(\delta) = (2p+1)\pi/k$. Notice that all the next results for the arcs $I(p, k, r, n)$ which will be showed in this paper can be proved by using $\arg(\delta) = 2\pi q/(n-k)$ where q is a nonzero integer.*

This family of arcs $I(p, k, r, n)$ (see the picture below) exists in view of the following lemma.



Trinomial arcs $I(p, k, r, n)$ inside the upper half unit disk

Lemma 3.6 *If n is an integer greater than or equal to* 3 *and $0 < \alpha < 1$, then in the trinomial equation* (1) *with the integer k verify $k = (2p+1)n/(2r+1)$, where p is an integer, r is a nonzero integer such that $r \geq p+1$, any angle of the interval $[2\pi r/n,\ (2p+1)\pi/k]$ is feasible.*

Proof. Let k be an integer satisfying $k = (2p+1)n/(2r+1)$. Let be $2\pi r/n < \theta < (2p+1)\pi/k$. It follows that $2\pi r < n\theta < (2r+1)\pi$ and that

$\sin n\theta > 0$. On the other side, we have $2\pi rk/n < k\theta < (2p+1)\pi$. Because $r \geq p+1$, we get $2\pi p < 2r(2p+1)\pi/(2r+1) = 2\pi rk/n$, so $\sin k\theta > 0$. Finally, we have $2\pi r(1-k/n) < (n-k)\theta < (2p+1)\pi(n/k-1)$. As $k = (2p+1)n/(2r+1)$, one has $4r(r-p)\pi/(2r+1) < (n-k)\theta < 2(r-p)\pi$. Since $[2(r-p)-1]\pi < 4r(r-p)\pi/(2r+1)$, then $\sin(n-k)\theta < 0$. The conditions (6) are so fulfilled.

Remark 3.7 *From the proof of Lemma 3.6, for each trinomial arc $I(p,k,r,n)$, we have $\sin n\theta > 0$, $\sin k\theta > 0$ and $\sin(n-k)\theta < 0$ for any θ in the interval $]2\pi r/n,\ (2p+1)\pi/k[$.*

4 Derivability of the function $\rho(\theta)$ for the arcs $I(p,k,r,n)$

Now, we will prove that the derivative $d\rho/d\theta$ exists and it is well-defined for the trinomial arcs $I(p,k,r,n)$.

Proposition 4.1 *For each trinomial arc $I(p,k,r,n)$, the function $\rho(\theta)$ is derivable for any feasible angle in the interval $]2\pi r/n,\ (2p+1)\pi/k[$.*

Proof. Let $I(p,k,r,n)$ be a trinomial arc. By equation (3), we have $\rho^k(\theta) = (1-1/\alpha)\sin n\theta/\sin(n-k)\theta$. According to Remark 3.7, the feasible angles θ are such that $\sin n\theta > 0$ and $\sin(n-k)\theta < 0$. If we put $f(\theta) = (1-1/\alpha)\sin n\theta/\sin(n-k)\theta$ and as $0 < \alpha < 1$, the denominator of $f(\theta)$ is never zero. The function $f(\theta)$ is so well-defined. In addition, f is derivable and positive. So, the function $\rho(\theta) = [f(\theta)]^{1/k}$ is derivable. Therefore, its derivative $d\rho/d\theta$ exists and it is well-defined.

5 Monotonicity of the function $\rho(\theta)$ for the arcs $I(p,k,r,n)$

In this section, our main interest is to show that $\rho(\theta)$ is a monotonic function, i.e. that the derivative $d\rho/d\theta$ is never zero, for each trinomial arc $I(p,k,r,n)$. Then, in equation (4), differentiating both sides with respect to θ, we obtain

$$\Big[(n-k)\,\rho^{n-k-1}\sin n\theta \,-\, n\,\rho^{n-1}\sin(n-k)\theta\Big]\ d\rho/d\theta$$
$$= k\,\cos k\theta \,+\, (n-k)\,\rho^{n}\cos(n-k)\theta \,-\, n\,\rho^{n-k}\cos n\theta.$$

Supposing that $d\rho/d\theta = 0$, we will consider ρ^n and ρ^{n-k} as solutions of the system :

$$\begin{cases} k\ \cos k\theta + (n-k)\ \rho^n\ \cos(n-k)\theta - n\ \rho^{n-k}\ \cos n\theta \ = \ 0 \\ \rho^{n-k}\ \sin n\theta - \rho^n\ \sin(n-k)\theta - \sin k\theta \ = \ 0 \end{cases}.$$

This system is equivalent to the following system :

$$\left\{ \begin{array}{rcl} R(\theta)\cdot\rho^{n-k} & = & N_1(\theta) \\ R(\theta)\cdot\rho^{n} & = & N_2(\theta) \end{array} \right. \tag{7}$$

where

$$\begin{array}{rcl} R(\theta) & = & (n-k)\ \sin k\theta\ -\ k\ \cos n\theta\ \sin(n-k)\theta \\ N_1(\theta) & = & (n-k)\ \sin n\theta\ -\ n\ \sin(n-k)\theta\ \cos k\theta \\ N_2(\theta) & = & (n-k)\ \sin n\theta\ \cos k\theta\ -\ n\ \sin(n-k)\theta. \end{array}$$

The difference of the two equalities of (7) leads to the equation :

$$R(\theta)\ [\rho^n - \rho^{n-k}] = U(\theta)\ [1 - \cos k\theta] \tag{8}$$

with

$$U(\theta) = -[n\ \sin(n-k)\theta\ +\ (n-k)\ \sin n\theta].$$

In what follows, the question is to contradict the hypothesis $d\rho/d\theta = 0$ for the family of trinomial arcs $I(p,k,r,n)$. For that, we need the two following lemmas.

Lemma 5.1 *For any integer k such that $k = (2p+1)n/(2r+1)$, we have $R(\theta) = (n-k)\sin k\theta - k\sin(n-k)\theta\cos n\theta > 0$ for any feasible angle θ in the interval $]2\pi r/n,\ (2p+1)\pi/k[$.*

Remark 5.2 *For the feasible angles θ, we have $2\pi r < n\theta < (2r+1)\pi$. Then, $\cos n\theta = 0$ if and only if $\theta = (4r+1)\pi/2n$. Moreover, we have $\cos n\theta > 0$ for $\theta < (4r+1)\pi/2n$ and $\cos n\theta < 0$ for $\theta > (4r+1)\pi/2n$.*

Proof. Let θ be a feasible angle in the interval $]2\pi r/n,\ (2p+1)\pi/k[$, where k is an integer verifying $k = (2p+1)n/(2r+1)$. By Remark 3.7, we have $\sin k\theta > 0$ and $\sin(n-k)\theta < 0$. From Remark 5.2, we get $R(\theta) > 0$ for any θ in $]2\pi r/n,\ (4r+1)\pi/2n]$. In the other case, i.e. when θ belongs to $](4r+1)\pi/2n,\ (2p+1)\pi/k[$, remarking that $R(\theta)$ can be expressed as $R(\theta) = (n-k)\sin n\theta\cos(n-k)\theta - n\sin(n-k)\theta\cos n\theta$, we will consider the function $K(\theta) = R(\theta)/\cos n\theta\cos(n-k)\theta = (n-k)\tan n\theta - n\tan(n-k)\theta$. In this case, we have $\cos n\theta < 0$. In addition, we have $(2r+1/2)\pi(1-k/n) < (n-k)\theta < (2p+1)\pi(n/k-1)$. As $k = (2p+1)n/(2r+1)$, one gets $2(2r+1/2)(r-p)\pi/(2r+1) < (n-k)\theta < 2(r-p)\pi$. Because $[2(r-p)-1/2]\,\pi < 2(2r+1/2)(r-p)\pi/(2r+1)$, we obtain that $\cos(n-k)\theta > 0$. The sign of $R(\theta)$ is so opposed to the sign of $K(\theta)$, which is derivable with $K'(\theta) = n(n-k)\,[\tan^2 n\theta - \tan^2(n-k)\theta]$. Since $\tan n\theta < 0$ and $\tan(n-k)\theta < 0$, the zeros of $K'(\theta)$ verify the equation $\tan n\theta = \tan(n-k)\theta$. Therefore, the unique solution of this equation is of the form $\theta = l\pi/k$ where l is an integer. However, $l\pi/k \in\ ](4r+1)\pi/2n,\ (2p+1)\pi/k[$ if and only if $(2r+1/2)k/n < l < (2p+1)$. As $k = (2p+1)n/(2r+1)$ and $r > p$, we get $(2p+1/2) < (2r+1/2)(2p+$

$1)/(2r+1) = (2r+1/2)k/n$. We have so $(2p+1/2) < l < (2p+1)$, which is not possible because l is an integer. We conclude that $K'(\theta)$ is never zero. Moreover, $K(\theta)$ goes to $-\infty$ as θ tends on the right to $(4r+1)\pi/2n$ and $K((2p+1)\pi/k) = 0$. It follows that $K(\theta) < 0$ and that $R(\theta) > 0$ for any θ in $](4r+1)\pi/2n,\ (2p+1)\pi/k[$. Therefore, $R(\theta) > 0$ for any feasible angle θ in the interval $]2\pi r/n,\ (2p+1)\pi/k[$.

Lemma 5.3 *For any integer k such that $k = (2p+1)n/(2r+1)$, we have $U(\theta) = -[n \sin(n-k)\theta + (n-k)\sin n\theta] > 0$ for any feasible angle θ in the interval $]2\pi r/n,\ (2p+1)\pi/k[$.*

Proof. Let θ be an angle in $]2\pi r/n,\ (2p+1)\pi/k[$, where the integer k is such that $k = (2p+1)n/(2r+1)$. The function $U(\theta)$ is derivable, with $U'(\theta) = -n(n-k)[\cos(n-k)\theta + \cos n\theta]$. The zeros of $U'(\theta)$ are of the form $\theta = (2l-1)\pi/k$ or of the form $\theta = (2l+1)\pi/(2n-k)$ where l is an integer. However, $(2l-1)\pi/k \in\ ]2\pi r/n,\ (2p+1)\pi/k[$ if and only if $rk/n + 1/2 < l < (p+1)$. As $r > p$, we obtain that $(p+1/2) < rk/n + 1/2$. Then, $(p+1/2) < l < (p+1)$, which is impossible as l is an integer. On the other side, $(2l+1)\pi/(2n-k) \in\ ]2\pi r/n,\ (2p+1)\pi/k[$ if and only if $2r(1-k/2n) - 1/2 < l < (2p+1)(n/k - 1/2) - 1/2$, i.e. $[r(4r-2p+1)/(2r+1)] - 1/2 < l < (2r-p)$. But $(2r-p-1) < [r(4r-2p+1)/(2r+1)] - 1/2$, which is not possible. It follows that $U'(\theta)$ is never zero. In addition, because $U(2\pi r/n) > 0$ and $U((2p+1)\pi/k) = 0$, we deduce that $U(\theta) > 0$ for any angle θ in $]2\pi r/n,\ (2p+1)\pi/k[$.

Thus, by using the two lemmas above, we can prove the next main result for the trinomial arcs $I(p,k,r,n)$.

Theorem 5.4 *The function $\rho(\theta)$ is monotonic on the interval of feasible angles $[2\pi r/n,\ (2p+1)\pi/k]$, for the trinomial arcs $I(p,k,r,n)$.*

Proof. Consider an arc $I(p,k,r,n)$. From Lemmas 5.1 and 5.3 stems respectively that $R(\theta) > 0$ and $U(\theta) > 0$ for any θ in $]2\pi r/n,\ (2p+1)\pi/k[$. Therefore, the relation $R(\theta)[\rho^n - \rho^{n-k}] = U(\theta)[1 - \cos k\theta]$ given by (8) implies that $\rho^n - \rho^{n-k} > 0$, which is impossible as $\rho < 1$. We have so proved that for each trinomial arc $I(p,k,r,n)$, we have $d\rho/d\theta \neq 0$, i.e. $\rho(\theta)$ is a monotonic function, for any angle θ in $]2\pi r/n,\ (2p+1)\pi/k[$. Thus, we achieve the proof.

In the end, Theorem 5.4 allows us to state the following main result.

Theorem 5.5 *$\rho(\theta)$ is a decreasing function on the interval of feasible angles $[2\pi r/n,\ (2p+1)\pi/k]$, for the trinomial arcs $I(p,k,r,n)$.*

Proof. Let $I(p,k,r,n)$ be a trinomial arc. According to Theorem 5.4, the function $\rho(\theta)$ is monotonic on $[2\pi r/n,\ (2p+1)\pi/k]$. Moreover, if we put $\theta = 2\pi r/n$ in the equation $\rho^{n-k}\sin n\theta - \rho^n \sin(n-k)\theta - \sin k\theta = 0$ given by (4), we get $(\rho^n - 1)\sin(2\pi rk/n) = 0$. As $k = (2p+1)n/(2r+1)$ and $r > p$, one has $2\pi p < 2\pi rk/n < (2p+1)\pi$, then $\sin(2\pi rk/n) \neq 0$. It follows that $\rho(2\pi r/n) = 1$. Since $\rho(\theta)$ is less than or equal to 1 for any feasible angle θ, we deduce that $\rho(\theta)$ is a decreasing function on the interval $[2\pi r/n,\ (2p+1)\pi/k]$.

6 Conclusion

In this work, we have studied the behavior of the family of trinomial arcs $I(p,k,r,n)$, composed of all solutions of equation (1) in the case $0 < \alpha < 1$ with the feasible angles θ in the interval $[\arg(\gamma), \arg(\delta)]$, where γ is an n^{th} root of unity and δ is both an k^{th} root of -1 and an $(n-k)^{th}$ root of unity. The problem of monotonicity of the trinomial arcs is completely solved in this case. During the description and definition of $I(p,k,r,n)$, we have evoked an other type of trinomial arcs, defined as the solutions of (1) in the case $0 < \alpha < 1$ with the feasible angles θ in the interval $[\arg(\delta), \arg(\gamma')]$, where δ is both an k^{th} root of -1 and an $(n-k)^{th}$ root of unity and γ' is an n^{th} root of unity. A later study of the behavior of this family of arcs would be interesting.

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