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Integers of the Form $p_1^{s_1} p_2^{s_2} \dots p_k^{s_k}$ where $p_1, p_2, \dots p_k$ are Primes Fixed

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Abstract

Let us consider the sequence A_n of all positive integers whose factorization is of the form $p_1^{s_1} p_2^{s_2} \dots p_k^{s_k}$ where $s_i \geq 0 \quad (i = 1, 2, \dots, k)$ and $p_1, p_2, \dots p_k$ are distinct primes fixed. Let $\psi(x)$ denote the number of these integers not exceeding x. We prove the following well known general result

$$\psi(x) = \frac{\ln^k x}{k! \ln p_1 \dots \ln p_k} + \frac{1}{(k-1)!} \frac{\ln \sqrt{p_1 \dots p_k}}{\ln p_1 \dots \ln p_k} \ln^{k-1} x + o(\ln^{k-1} x)$$

The result when $k = 2$ was obtained by Ramanujan (letter to Hardy), Hardy and Littlewood, and others (see [2], chapter V, and [3]).

D. H. Lehmer [4] was the first to consider the n-dimensional analogue of the 2-dimensional problem considered by Ramanujan, Hardy and Littlewood, and others.

D. C. Spencer [5], using complex function-theoretic methods, and F. Beukers [1], using elementary methods, obtained this general result.

In this article we obtain this general result in a very elementary and short form. We use mathematical induction (as F. Beukers in his paper) and combinatory. We also prove

$$A_n \sim \frac{1}{\sqrt{p_1 \cdots p_k}} exp(\sqrt[k]{k! \ln p_1 \dots \ln p_k n})$$

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1 Preliminary results

Let us consider the linear inequality

$$r_1 x_1 + r_2 x_2 + \dots + r_n x_n \leq x \qquad (x \geq 0) \tag{1}$$

Where the numbers $r_i > 0 \quad (i = 1, 2, \ldots, n)$ and $n \geq 2$ are fixed.

Let $S_n(x)$ be the number of solutions $(x_1, x_2, \ldots, x_n)$ to the inequality (1) where the $x_j \quad (j = 1, 2, \ldots, n)$ are positive integers.

The following result can be proved without difficulty by mathematical induction.

Lemma 1.1 *The following formula holds*

$$S_n(x) = \frac{1}{n!}\frac{x^n}{r_1 r_2 \ldots r_n} + f_n(x)x^{n-1} \qquad (x \geq 0)$$

Where $|f_n(x)| < K_n$ *in the interval* $[0, \infty)$, K_n *being a certain positive number.*

The following theorem is a simple consequence of the results exposed in [2] (chapter V). We shall use it as a fundamental lemma.

Lemma 1.2 *Let us consider the inequality (1) when* $n = 2$. *If* r_1/r_2 *is an irrational number, then the following formula holds*

$$S_2(x) = \frac{1}{2}\frac{x^2}{r_1 r_2} - \frac{1}{2}\frac{r_1 + r_2}{r_1 r_2}x + o(x) \qquad (x \geq 0)$$

Now, it is well known if s is a nonnegative integer

$$\sum_{i=0}^{K} i^s = \sum_{i=0}^{s+1} a_{i,s}K^i \quad (s = 0) \quad (0^0 = 1), \qquad \sum_{i=0}^{K} i^s = \sum_{i=1}^{s+1} a_{i,s}K^i \quad (s \geq 1)$$

If $s \geq 1$ the first coefficient is $a_{s+1,s} = 1/(s+1)$ and the second coefficient is $a_{s,s} = 1/2$. There are many elementary proofs on this subject. For example, mathematical induction.

Lemma 1.3 *If* $M \neq 0$ *and* r *are real numbers then*

$$\sum_{i=0}^{K}(Mi + r)^s = \sum_{i=0}^{s+1} A_{i,s}(MK + r)^i \quad (s \geq 0)$$

where $A_{0,s} = r^s - \sum_{i=1}^{s+1} A_{i,s}r^i$ *and* $A_{i,s} = a_{i,s}M^{s-i} \quad (i = 1, 2, \ldots, s + 1)$

Remark 1. Note if $s \geq 1$ we find that the first coefficient is $A_{s+1,s} = 1/(s + 1)M$ and the second coefficient is $A_{s,s} = 1/2$.

Example 1.4 *We have*

$$\sum_{i=0}^{K} i^2 = \frac{1}{3}K^3 + \frac{1}{2}K^2 + \frac{1}{6}K$$

Hence

$$\begin{aligned}\sum_{i=0}^{K}(Mi + r)^2 &= \frac{1}{3M}(MK + r)^3 + \frac{1}{2}(MK + r)^2 + \frac{M}{6}(MK + r) \\ &+ \left(r^2 - \frac{1}{3M}r^3 - \frac{1}{2}r^2 - \frac{M}{6}r\right)\end{aligned}$$

Proof. We proceed by mathematical induction. If $s = 0$ the theorem is clearly true. Suppose the theorem is true for $0, 1, \dots, s-1 \quad (s \geq 1)$. We shall prove it is true for s. If $s \geq 1$ we have

$$(M(i-1)+r)^{s+1} = \left(\sum_{t=0}^{s} \binom{s+1}{t} (Mi+r)^t (-M)^{s+1-t} \right) + (Mi+r)^{s+1}$$

Summing over $0 \leq i \leq K$ we get

$$\begin{aligned} \sum_{i=0}^{K} (Mi+r)^s &= \frac{(MK+r)^{s+1}}{M(s+1)} - \left(\sum_{t=0}^{s-1} \frac{1}{s+1} \binom{s+1}{t} (-M)^{s-t} \sum_{i=0}^{K} (Mi+r)^t \right) \\ &- \frac{(r-M)^{s+1}}{M(s+1)} \end{aligned} \qquad (2)$$

Equation (2) gives

$$\sum_{i=0}^{K} (Mi+r)^s = \sum_{i=0}^{s+1} A_{i,s} (MK+r)^i \qquad (3)$$

In equation (3) we have: 1) The left side is a polynomial in K. 2) The right side is a polynomial in K. 3) Both polynomials are equal. 4) The coefficients $A_{i,s}$ do not depend of r (i=1, 2, ..., s+1).

If $r = 0$ equation (3) becomes

$$\sum_{i=0}^{K} (Mi)^s = M^s \sum_{i=0}^{K} i^s = \sum_{i=1}^{s+1} a_{i,s} M^s K^i = \sum_{i=1}^{s+1} A_{i,s} (MK)^i = \sum_{i=1}^{s+1} A_{i,s} M^i K^i$$

Therefore $A_{i,s} = a_{i,s} M^{s-i} \quad (i = 1, 2, \dots, s+1)$.

If $K = 0$ in equation (3) we find that $\sum_{i=0}^{K} (Mi+r)^s = r^s = A_{0,s} + \sum_{i=1}^{s+1} A_{i,s} r^i$. Lemma 1.3 is proved.

2 The main theorem

Theorem 2.1 *Let us consider the inequality (1). If* r_1/r_2 *is an irrational number, then*

$$S_n(x) = \frac{1}{n!} \frac{x^n}{r_1 \dots r_n} - \frac{1}{2(n-1)!} \frac{r_1 + \dots + r_n}{r_1 \dots r_n} x^{n-1} + o(x^{n-1}) \qquad (x \geq 0) \quad (4)$$

Proof. We proceed by mathematical induction. If $n = 2$ the theorem is true (lemma 1.2). Consider the inequality (1), that is

$$r_1 x_1 + \dots + r_{n-1} x_{n-1} + r_n x_n \leq x \qquad (x \geq 0)$$

Suppose the theorem is true for $n-1 \quad (n \geq 3)$. We shall prove it is also true for n.

If we consider the inequality

$$r_1 x_1 + \ldots + r_{n-1} x_{n-1} \leq a \qquad (a \geq 0)$$

then (inductive hypothesis)

$$S_{n-1}(a) = \frac{1}{(n-1)!} \frac{a^{n-1}}{r_1 \ldots r_{n-1}} + f_{n-1}(a) a^{n-2} \tag{5}$$

where $|f_{n-1}(a)| < K_{n-1}$ in the interval $[0, \infty)$ (lemma 1.1), and the following limit holds

$$\lim_{a \to \infty} f_{n-1}(a) = -\frac{1}{2(n-2)!} \frac{r_1 + \ldots + r_{n-1}}{r_1 \ldots r_{n-1}} = L \tag{6}$$

We have from the inductive hypothesis the number of solutions to the inequality (1) will be (for sake of simplicity in the notation we shall write $b = x - r_n x_n$)

$$S_n(x) = \sum_{x_n=1}^{[x/r_n]} S_{n-1}(b) = \frac{1}{(n-1)! r_1 \ldots r_{n-1}} \sum_{x_n=1}^{[x/r_n]} b^{n-1} + \sum_{x_n=1}^{[x/r_n]} f_{n-1}(b) b^{n-2} \tag{7}$$

The function $f(x_n) = b^{n-1}$ is decreasing in the interval $0 \leq x_n \leq x/r_n$, besides $f(0) = x^{n-1}$ and $f(x/r_n) = 0$. Therefore we have

$$\begin{aligned} \frac{1}{(n-1)! r_1 \ldots r_{n-1}} \sum_{x_n=1}^{[x/r_n]} b^{n-1} &= \frac{1}{(n-1)! r_1 \ldots r_{n-1}} \left(\int_0^{x/r_n} b^{n-1} dx_n - G_1(x) \right) \\ &= \frac{x^n}{n! r_1 \ldots r_n} - \frac{1}{(n-1)! r_1 \ldots r_{n-1}} G_1(x) \end{aligned} \tag{8}$$

where $0 \leq G_1(x) \leq x^{n-1}$. In the same form we obtain

$$\sum_{x_n=1}^{[x/r_n]} b^{n-2} = \int_0^{x/r_n} b^{n-2} dx_n - G_2(x) = \frac{x^{n-1}}{(n-1) r_n} - G_2(x) \tag{9}$$

where $0 \leq G_2(x) \leq x^{n-2}$. Hence $G_2(x) = o(x^{n-1})$.

Equations (4), (7) and (8) give

$$f_n(x) = \frac{1}{x^{n-1}} \left(-\frac{1}{(n-1)! r_1 \ldots r_{n-1}} G_1(x) + \sum_{x_n=1}^{[x/r_n]} f_{n-1}(b) b^{n-2} \right) \tag{10}$$

If $s = n-1 \quad (n \geq 3)$, $M = -r_n$, $r = x$ and $K = [x/r_n]$, lemma 1.3 gives

$$G_1(x) = \frac{x^n}{n r_n} - \sum_{x_n=1}^{[x/r_n]} b^{n-1} = \frac{1}{2} x^{n-1} + o\left(x^{n-1}\right) \tag{11}$$

Let $\epsilon > 0$ be, then there exists x_0 such that if $x \geq x_0$ we have (see equation (6))

$$L - \epsilon < f_{n-1}(x) < L + \epsilon \tag{12}$$

The inequality $b \geq x_0$ is true if and only if $x_n = 1, 2, \dots, [(x - x_0)/r_n]$.

We have

$$\sum_{x_n=[(x-x_0)/r_n]+1}^{[x/r_n]} b^{n-2} \leq ([x/r_n] - [(x - x_0)/r_n])\, x_0^{n-2} \leq ((x_0/r_n) + 1)\, x_0^{n-2} \tag{13}$$

and

$$\left| \sum_{x_n=[(x-x_0)/r_n]+1}^{[x/r_n]} f_{n-1}(b) b^{n-2} \right| \leq K_{n-1}((x_0/r_n) + 1)\, x_0^{n-2} \tag{14}$$

Equation (12) gives

$$(L - \epsilon) \sum_{x_n=1}^{[(x-x_0)/r_n]} b^{n-2} \leq \sum_{x_n=1}^{[(x-x_0)/r_n]} f_{n-1}(b) b^{n-2} \leq (L + \epsilon) \sum_{x_n=1}^{[(x-x_0)/r_n]} b^{n-2} \tag{15}$$

Equations (9) and (13) give

$$\sum_{x_n=1}^{[(x-x_0)/r_n]} b^{n-2} = \frac{x^{n-1}}{(n-1)r_n} - G_2(x) + O(1) = \frac{x^{n-1}}{(n-1)r_n} + o\left(x^{n-1}\right) \tag{16}$$

Equations (15) and (16) give (x large)

$$\begin{aligned} \frac{L}{(n-1)r_n} - \frac{\epsilon}{(n-1)r_n} - \epsilon \quad &\leq \quad \frac{\sum_{x_n=1}^{[(x-x_0)/r_n]} f_{n-1}(b) b^{n-2}}{x^{n-1}} \\ &\leq \quad \frac{L}{(n-1)r_n} + \frac{\epsilon}{(n-1)r_n} + \epsilon \end{aligned} \tag{17}$$

Finally equations (10), (11), (14), (17) and (6) give

$$\lim_{x \to \infty} f_n(x) = -\frac{1}{2(n-1)!r_1 \dots r_{n-1}} + \frac{L}{(n-1)r_n} = -\frac{1}{2(n-1)!} \frac{r_1 + \dots + r_n}{r_1 \dots r_n}$$

Theorem 2.1 is proved.

Let $S'_n(x)$ be the number of solutions $(x_1, \dots, x_n)$ to the inequality (1) where the $x_j \quad (j = 1, ..., n)$ are nonnegative integers. From theorem 2.1 we obtain without difficulty the following,

Corollary 2.2 *Let us consider the inequality (1). If* r_1/r_2 *is an irrational number, then*

$$S'_n(x) = \frac{1}{n!} \frac{x^n}{r_1 \dots r_n} + \frac{1}{2(n-1)!} \frac{r_1 + \dots + r_n}{r_1 \dots r_n} x^{n-1} + o(x^{n-1}) \qquad (x > 0)$$

3 Integers of the form $p_1^{s_1}\dots p_k^{s_k}$

Let us consider the sequence A_n of all positive integers whose factorization is of the form $p_1^{s_1}\dots p_k^{s_k}$ where $s_i \geq 0 \quad (i = 1, 2, \dots, k)$ and $p_1, \dots p_k \quad (k \geq 2)$ are distinct primes fixed. Let $\psi(x)$ denote the number of these integers not exceeding x. A direct consequence of corollary 2.2 is the following theorem,

Theorem 3.1 *The following formula holds*

$$\psi(x) = \frac{\ln^k x}{k! \ln p_1 \dots \ln p_k} + \frac{1}{(k-1)!} \frac{\ln \sqrt{p_1 \dots p_k}}{\ln p_1 \dots \ln p_k} \ln^{k-1} x + o(\ln^{k-1} x) \qquad (18)$$

Where $x > 1$.

From theorem 3.1 we obtain without difficulty the following corollary,

Corollary 3.2 *If $h > 1$ the following asymptotic formula holds*

$$\psi(hx) - \psi(x) \sim \frac{\ln h}{(k-1)! \ln p_1 \dots \ln p_k} \ln^{k-1} x$$

Theorem 3.3 *The following asymptotic formula holds*

$$A_n \sim \frac{1}{\sqrt{p_1 \cdots p_k}} exp\left(\sqrt[k]{k! \ln p_1 \dots \ln p_k n}\right.$$

Proof. Substituting $x = A_n$ into (18) we obtain

$$\begin{aligned} k! \ln p_1 \dots \ln p_k n &= \ln^k A_n + k \ln \sqrt{p_1 \dots p_k} \ln^{k-1} A_n + o\left(\ln^{k-1} A_n\right) \\ &= (\ln A_n + \ln \sqrt{p_1 \dots p_k})^k + o\left(\ln^{k-1} A_n\right) \end{aligned}$$

That is

$$\begin{aligned} \sqrt[k]{k! \ln p_1 \dots \ln p_k n} &= \ln\left(\sqrt{p_1 \dots p_k} A_n\right) \sqrt[k]{1 + o\left(\frac{1}{\ln A_n}\right)} \\ &= \ln\left(\sqrt{p_1 \dots p_k} A_n\right) \left(1 + o\left(\frac{1}{\ln A_n}\right)\right) \end{aligned}$$

The theorem is proved.

Corollary 3.4 *The following limit holds*

$$\lim_{n \to \infty} \frac{A_{n+1}}{A_n} = 1$$

Remark 2. a) We can consider the subsequence of all numbers whose factorization is of the form $p_1^{c_1 s_1} p_2^{c_2 s_2} \dots p_k^{c_k s_k}$ where $c_1, c_2, \dots c_k$ are positive integer fixed. The section 3 can be rewritten for this subsequence. b) Let us consider the sequence B_n of all numbers whose factorization is of the form $p_1^{s_1} p_2^{s_2} \dots p_k^{s_k}$ where $s_i > 0 \quad (i = 1, 2, \dots, k)$ and $p_1, p_2, \dots p_k$ $(k \geq 2)$ are distinct primes fixed. The section 3 can be rewritten for this sequence using theorem 2.1.

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