Properties of Convolutions
Arising in Stochastic Volterra Equations

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Abstract

The aim of this paper is to provide some results for stochastic convolutions corresponding to stochastic Volterra equations in separable Hilbert space. We study convolution of the form $W^{\Psi}(t) := \int_0^t S(t-\tau)\Psi(\tau)dW(\tau)$, $t \geq 0$, where $S(t)$, $t \geq 0$, is so-called `resolvent` for Volterra equation considered, Ψ is an appropriate process and W is a cylindrical Wiener process.

1 Definitions and notation

In the paper we consider the following stochastic Volterra equation in a separable Hilbert space H:

$$X(t) = X(0) + \int_0^t a(t-\tau)AX(\tau)\,d\tau + \int_0^t \Psi(\tau)\,dW(\tau), \qquad (1)$$

where $t \in \mathbb{R}_+$, $a \in L^1_{\text{loc}}(\mathbb{R}_+)$, A is a closed unbounded linear operator in H with a dense domain D(A), Ψ is an adapted integrable stochastic process specified below, W is a cylindrical Wiener process with respect to t and $X(0)$ belongs to H.

The stochastic Volterra equations have been studied in connection with applications to problems arising in mathematical physics, particularly in viscoelasticity and heat conduction in materials with memory. We refer to the papers [3], [4] and [5]. Let us note that the equation (1) is a generalization of stochastic heat and wave equations and stochastic linear Navier-Stokes system.

The above equation (1) is a stochastic version of the deterministic Volterra equation of the form

$$u(t) = \int_0^t a(t - \tau)Au(\tau)d\tau + f(t), \tag{2}$$

where elements in (2) are the same as in (1), and f is an appropriate H-valued mapping.

By $S(t)$, $t \geq 0$, we shall denote the family of resolvent operators corresponding to the Volterra equation (2) and defined as follows.

Definition 1 *(see, e.g. [13])*
A family $(S(t))_{t \geq 0}$ of bounded linear operators in the space H is called `resolvent` *for (2) if the following conditions are satisfied:*

1. *$S(t)$ is strongly continuous on $\mathbb{R}_+$ and $S(0) = I$;*

2. *$S(t)$ commutes with the operator A, that is, $S(t)(D(A)) \subset D(A)$ and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and $t \geq 0$;*

3. *the following* `resolvent equation` *holds*

$$S(t)x = x + \int_0^t a(t - \tau)AS(\tau)x d\tau \tag{3}$$

for all $x \in D(A)$, $t \geq 0$.

Proposition 1 *([13, Proposition 1.1]) The equation (2) is well-posed if and only if (2) admits a resolvent $S(t)$. If this is the case then, in addition, $R(a \star S(t)) \subset D(A)$ for all $t \geq 0$ and*

$$S(t)x = x + A\int_0^t a(t - \tau)S(\tau)x d\tau \quad \text{for all} \quad x \in H, \ t \geq 0. \tag{4}$$

Comment: Let us emphasize that the resolvent $S(t)$, $t \geq 0$, is determined by the operator A and the function a. Moreover, as a consequence of the strong continuity of $S(t)$ we have for any $T > 0$

$$\sup_{t \leq T} ||S(t)|| < +\infty , \tag{5}$$

where $|| \cdot ||$ denotes the operator norm.

In the paper we shall assume that the equation (2) is *well-posed* what means that (2) admits a resolvent $S(t)$.

The so-called resolvent approach to the Volterra equation (2) has been introduced many years ago, probably by Friedman and Shinbrot [8], but recently has been presented in details in the great monograph by Prüss [13]. The resolvent approach is a generalization of the semigroup approach. In the consequence, problems concerning convolutions with resolvents (defined below by (8)) are more difficult than in previous case because of lack of the semigroup property.

The main aim of the paper is to provide some introductory regularity results for stochastic convolutions with resolvent operators, analogous to that obtained in e.g. [6], [14] and [7], that is, to extend semigroup approach for our, non-semigroup case.

In order to make the paper self-contained, we formulate definitions and auxiliary lemmas necessary for understanding the main results.

Assume that $(\Omega, \mathcal{F}, P)$ is a probability space equipped with an increasing family of σ-fields $(\mathcal{F}_t)$, $t \in [0, T]$ called *filtration*. We shall denote by $\mathcal{F}_{t^+}$ the intersection of all σ-fields $\mathcal{F}_s$, $s > t$. We say that filtration is *normal* if $\mathcal{F}_0$ contains all sets $B \in \mathcal{F}$ with measure $P(B) = 0$ and if $\mathcal{F}_t = \mathcal{F}_{t^+}$ for any $t \in I$, that is, the filtration is right continuous.

In the paper we assume that filtration $(\mathcal{F}_t)_{t \in I}$ is normal. This assumption enables to choose modifications of considered stochastic processes with required measurable properties.

We will use the following well-known result, see e.g. [7].

Proposition 2 *Let $X(t)$, $t \in [0, T]$, be a stochastically continuous and adapted process with values in H. Then X has a progressively measurable modification.*

In the paper stochastic processes Ψ and W are defined as follows. We consider two separable Hilbert spaces H and U and a Wiener process W on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$, having values in some superspace of U with the nonnegative covariance operator $Q \in L(U)$. (By $L(U, H)$, $L(U)$ we denote spaces of linear bounded operators from U into H and in U, respectively.) We assume that the process W is a cylindrical one, that is, we do not assume that $\text{Tr} \, Q < +\infty$. In this case, the process W has continuous paths in some other Hilbert space (for details, see [7] or [10]). Assume that there exists a complete orthonormal set $\{e_k\} \subset U$ of eigenvectors of the operator Q with corresponding eigenfunctions λ_k, $k = 1, 2, \ldots$; so $\text{Tr} Q = \sum_{k=1}^{\infty} \lambda_k$. We shall use the following expansion of the process $W(t) = \sum_{k=1}^{\infty} e_k \beta_k(t)$, where $\beta_k(t)$ are independent real Wiener processes with $\mathbb{E}(\beta_k^2(t)) = \lambda_k t$. We will need the subspace $U_0 := Q^{1/2}(U)$ of the space U, which endowed with the inner product $\langle u, v \rangle_{U_0} := \langle Q^{-1/2} u, Q^{-1/2} v \rangle_U$

forms a Hilbert space. Here and in the whole paper we write explicitly indexes indicating the appropriate space in norms $|\cdot|_{(\cdot)}$ and inner products $\langle\cdot,\cdot\rangle_{(\cdot)}$.

This is apparently well-known fact that the construction of the stochastic integral with respect to cylindrical Wiener process requires some particular terms. Among others, an important role is played by the space of Hilbert-Schmidt operators. A linear, bounded operator C acting from U_0 into H is called a *Hilbert-Schmidt* if $\sum_{k=1}^{+\infty}|Cu_k|_H^2 < +\infty$, where $\{u_k\}\subset U_0$ is an orthonormal base in U_0. The set $L_2(U_0,H)$ of all Hilbert-Schmidt operators from U_0 into H, equipped with the norm $|C|_{L_2(U_0,H)} := (\sum_{k=1}^{+\infty}|Cu_k|_H^2)^{1/2}$, is a separable Hilbert space. For abbreviation we denote $L_2^0 := L_2(U_0,H)$. (For more details concerning that space we refer to [2] or [7].)

Let $\Phi(t)$, $t\in[0,T]$, be a measurable L_2^0-valued process. We introduce the norms

$$
\begin{aligned}
||\Phi||_t &:= \left\{\mathbb{E}\left(\int_0^t|\Phi(\tau)|_{L_2^0}^2\,d\tau\right)\right\}^{\frac{1}{2}} \\
&= \left\{\mathbb{E}\int_0^t\left[\mathrm{Tr}(\Phi(\tau)Q^{\frac{1}{2}})(\Phi(\tau)Q^{\frac{1}{2}})^*\right]d\tau\right\}^{\frac{1}{2}}, \quad t\in[0,T].
\end{aligned}
$$

By $\mathcal{N}^2(0,T;L_2^0)$ we shall denote a Hilbert space of all L_2^0-predictable processes Φ such that $||\Phi||_T < +\infty$.

According to the theory of stochastic integral with respect to cylindrical Wiener process (see [7] or [10]) we have to assume that Ψ belongs to the space $\mathcal{N}^2(0,T;L_2^0)$. There is possible to consider a more general class of integrands, that is, the class of L_2^0-predictable processes satisfying condition $P\left(\int_0^T|\Psi(\tau)|_{L_2^0}^2\,d\tau < +\infty\right) = 1$. Such processes are called *stochastically integrable* on $[0,T]$ and create a linear space denoted by $\mathcal{N}(0,T;L_2^0)$. But, in our opinion, it is not worthwhile to study the general case, because this assumption makes all formulations of results much more complicated. Moreover, it produces a new level of difficulty additionally to problems related to long time memory of the system.

In the whole paper we shall use the following VOLTERRA ASSUMPTIONS (abbr. (VA)):

1. $A : D(A)\subset H\to H$, is a closed linear unbounded operator with the dense domain;

2. $a\in L_{\mathrm{loc}}^1(\mathbb{R}_+)$;

3. the equation (2) is well-posed and $S(t)$, $t\geq 0$, are resolvent operators for the Volterra equation (2) determined by the operator A and the function a.

For $h \in D(A)$ we define the graph norm as follows: $|h|_{D(A)} := (|h|_H^2 + |Ah|_H^2)^{\frac{1}{2}}$. Because H is a separable Hilbert space and A is a closed operator, the space $(D(A), | \cdot |_{D(A)})$ is a separable Hilbert space.

Moreover, we shall study the equation (1) under the following PROBABIL-ITY ASSUMPTIONS (abbr. (PA)):

1. $X(0)$ is an H-valued, $\mathcal{F}_0$-measurable random variable;

2. Ψ belongs to the space $\mathcal{N}^2(0, T; L_2^0)$, where the finite interval $[0, T]$ is fixed.

Now, we introduce the definitions of solutions to the stochastic Volterra equation (1).

Definition 2 *Assume that conditions (VA) and (PA) hold. An H-valued pre-dictable process $X(t)$, $t \in [0, T]$, is said to be a* `strong solution` *to (1), if X has a version such that $P(X(t) \in D(A)) = 1$ for almost all $t \in [0, T]$; for any $t \in [0, T]$, $\int_0^t |a(t - \tau)AX(\tau)|_H d\tau < +\infty$, P-a.s. and for any $t \in [0, T]$ the equation (1) holds P-a.s.*

Comment: Because the integral $\int_0^\bullet \Psi(\tau) \, dW(\tau)$ is a continuous H-valued process then the above definition yields continuity of the strong solution.

Let A^* denote the adjoint of the operator A, with dense domain $D(A^*) \subset H$ and the graph norm $| \cdot |_{D(A^*)}$ defined as follows: $|h|_{D(A^*)} := (|h|_H^2 + |A^*h|_H^2)^{\frac{1}{2}}$, for $h \in D(A^*)$. The space $(D(A^*), | \cdot |_{D(A^*)})$ is a separable Hilbert space.

Definition 3 *Let conditions (VA) and (PA) hold. An H-valued predictable process $X(t)$, $t \in [0, T]$, is said to be a* `weak solution` *to (1), if $P(\int_0^t |a(t - \tau)X(\tau)|_H d\tau < +\infty) = 1$ and if for all $\xi \in D(A^*)$ and all $t \in [0, T]$ the following equation holds*

$$\langle X(t), \xi \rangle_H = \langle X(0), \xi \rangle_H \; + \; \langle \int_0^t a(t - \tau)X(\tau) \, d\tau, A^*\xi \rangle_H$$
$$+ \; \langle \int_0^t \Psi(\tau) \, dW(\tau), \xi \rangle_H, \quad P-\text{a.s.}$$

Definition 4 *Assume that (VA) are satisfied and $X(0)$ is an H-valued $\mathcal{F}_0$-measurable random variable. An H-valued predictable process $X(t)$, $t \in [0, T]$, is said to be a* `mild solution` *to the stochastic Volterra equation (1), if*

$$\mathbb{E} \left(\int_0^t |S(t - \tau)\Psi(\tau)|_{L_2^0}^2 d\tau \right) < +\infty \quad for \quad t \leq T \tag{6}$$

and, for arbitrary $t \in [0, T]$,

$$X(t) = S(t)X(0) + \int_0^t S(t - \tau)\Psi(\tau) \, dW(\tau), \quad P - a.s. \tag{7}$$

In some cases weak solutions to the equation (1) coincide with mild solutions to (1). In consequence, having results for the convolution

$$W^{\Psi}(t) := \int_0^t S(t - \tau)\Psi(\tau) \, dW(\tau), \quad t \in [0, T], \tag{8}$$

where $S(t)$ and Ψ are the same as in (7), we obtain results for weak solution to (1).

2 Introductory results

In this section we collect some basic properties of the stochastic convolution of the form

$$W^B(t) := \int_0^t S(t - \tau)B \, dW(\tau) \tag{9}$$

in the case when $B \in L(U, H)$.

Lemma 1 *Assume that the operators $S(t)$, $t \geq 0$, and B are as above, $S^*(t)$, B^* are their adjoints, and*

$$\int_0^T |S(\tau)B|_{L_2^0}^2 \, d\tau = \int_0^T \mathrm{Tr}[S(\tau)BQB^*S^*(\tau)] \, d\tau < +\infty. \tag{10}$$

Then we have:

  (i) *the process W^B is Gaussian, mean-square continuous on [0,T] and then has a predictable version;*

  (ii)

$$\mathrm{Cov}\, W^B(t) = \int_0^t [S(\tau)BQB^*S^*(\tau)] \, d\tau, \quad t \in [0, T]; \tag{11}$$

  (iii) *trajectories of the process W^B are P-a.s. square integrable on [0,T].*

Proof:

(i) Gaussianity of the process W^B follows from the definition and properties of stochastic integral. Let us fix $0 \le t < t + h \le T$. Then

$$W^B(t + h) - W^B(t) = \int_0^t [S(t + h - \tau) - S(t - \tau)] B dW(\tau)$$
$$+ \int_t^{t+h} S(t + h - \tau) B dW(\tau).$$

Let us note that the above integrals are stochastically independent. Using the extension of the process W (mentioned in section 1) and properties of stochastic integral with respect to real Wiener processes (see, e.g., [9]), we have

$$\mathbb{E}|W^B(t + h) - W^B(t)|_H^2 = \sum_{k=1}^{+\infty} \lambda_k \int_0^t |[S(t+h-\tau) - S(t-\tau)] B e_k|_H^2 \, d\tau$$
$$+ \sum_{k=1}^{+\infty} \lambda_k \int_t^{t+h} |S(t + h - \tau) B e_k|_H^2 \, d\tau$$
$$:= I_1(t, h) + I_2(t, h) .$$

Then, invoking (5), the strong continuity of $S(t)$ and the Lebesgue dominated convergence theorem, we can pass in $I_1(t, h)$ with $h \to 0$ under the sum and integral signs. Hence, we obtain $I_1(t, h) \to 0$ as $h \to 0$.

Observe that

$$I_2(t, h) = \int_t^{t+h} ||S(t + h - \tau) B Q^{\frac{1}{2}}||_{HS}^2 \, d\tau ,$$

where $|| \cdot ||_{HS}$ denotes the norm of Hilbert-Schmidt operator. By the condition (10) we have

$$\int_0^T ||S(t) B Q^{\frac{1}{2}}||_{HS}^2 \, dt < +\infty ,$$

what follows that $\lim_{h \to 0} I_2(t, h) = 0$.

The proof for the case $0 \le t - h < t \le T$ is similar. Existence of a predictable version is a consequence of the above continuity and Proposition 2.

(ii) Covariance (11) follows from theory of stochastic integral.

(iii) From the definition (9) and assumption (10) we have the following estimate

$$\mathbb{E}\int_0^T |W^B(\tau)|_H^2\, d\tau = \int_0^T \mathbb{E}|W^B(\tau)|_H^2\, d\tau =$$

$$= \int_0^T \mathbb{E}\left|\int_0^\tau S(\tau-r)B dW(r)\right|_H^2 d\tau = \int_0^T\int_0^\tau |S(r)B|_{L_2^0}^2\, dr\, d\tau < +\infty\,.$$

Hence, the function $W^B(\cdot)$ may be regarded like random variable with values in the space $L^2(0,T;H)$. ∎

Comment: Let us emphasize that Clément and Da Prato (see [3] and [4]) obtained Hölderianity of the trajectories of the stochastic convolutions $W_{A,a}(t) := \int_0^t S(t-\tau)dW(\tau)$ in the case when A is a self-adjoint negative operator in H fulfilling some technical assumptions and when $a \in L_{\text{loc}}^1(\mathbb{R}_+)$ is a completely positive function. In that case the operator norm $||S(t)|| \in [0,1]$ for any $t \in [0,T]$.

Analogously like in the theory of evolution equation we can obtain the following result.

Theorem 1 *Assume that the operators $S(t)$, $t \geq 0$, and B are as above and the condition (10) holds. Let $X(0)$ be a $\mathcal{F}_0$-measurable random variable with values in $D(A)$. Then the stochastic Volterra equation (1) has exactly one weak solution which is given by the formula*

$$X(t) = S(t)X(0) + \int_0^t S(t-\tau)B dW(\tau).$$

Now, we formulate an auxiliary result which will be used in the next section.

Lemma 2 *Let* VOLTERRA ASSUMPTIONS *hold with the kernel function $a \in W^{1,1}(\mathbb{R}_+)$. Assume that X is a weak solution to (1) in the case when $\Psi(t) = B$, where $B \in L(U,H)$ and trajectories of X are integrable w.p. 1 on $[0,T]$. Then, for any function $\xi \in C^1([0,t];D(A^*))$, $t \in [0,T]$, the following formula holds*

$$\langle X(t), \xi(t)\rangle_H = \langle X(0), \xi(0)\rangle_H + \int_0^t \langle (\dot{a}\star X)(\tau) + a(0)X(\tau), A^*\xi(\tau)\rangle_H d\tau$$

$$+ \int_0^t \langle \xi(\tau), B dW(\tau)\rangle_H + \int_0^t \langle X(\tau), \dot{\xi}(\tau)\rangle_H d\tau, \qquad (12)$$

where dots above a and ξ mean time derivatives and $\star$ means the convolution.

Proof: First, we consider functions of the form $\xi(\tau) := \xi_0 \varphi(\tau)$, $\tau \in [0, T]$, where $\xi_0 \in D(A^*)$ and $\varphi \in C^1[0, T]$. For simplicity we omit index $_H$ in the inner product. Let us denote $F_{\xi_0}(t) := \langle X(t), \xi_0 \rangle$, $t \in [0, T]$.

Using Itô's formula to the process $F_{\xi_0}(t)\varphi(t)$, we have

$$d[F_{\xi_0}(t)\varphi(t)] = \varphi(t)dF_{\xi_0}(t) + \dot{\varphi}(t)F_{\xi_0}(t)dt, \qquad t \in [0, T]. \tag{13}$$

Because X is weak solution to (1), we have

$$
\begin{aligned}
dF_{\xi_0}(t) &= \langle \int_0^t \dot{a}(t - \tau)X(\tau)d\tau + a(0)X(t), A^*\xi_0 \rangle dt + \langle BdW(t), \xi_0 \rangle \\
&= \langle (\dot{a} \star X)(t) + a(0)X(t), A^*\xi_0 \rangle dt + \langle BdW(t), \xi_0 \rangle. \tag{14}
\end{aligned}
$$

From (13) and (14), we obtain

$$
\begin{aligned}
F_{\xi_0}(t)\varphi(t) &= F_{\xi_0}(0)\varphi(0) + \int_0^t \varphi(s)\langle (\dot{a} \star X)(s) + a(0)X(s), A^*\xi_0 \rangle ds \\
&\quad + \int_0^t \langle \varphi(s)BdW(s), \xi_0 \rangle + \int_0^t \dot{\varphi}(s)\langle X(s), \xi_0 \rangle ds \\
&= \langle X(0), \xi(0) \rangle_H + \int_0^t \langle (\dot{a} \star X)(s) + a(0)X(s), A^*\xi(s) \rangle ds \\
&\quad + \int_0^t \langle BdW(s), \xi(s) \rangle + \int_0^t \langle X(s), \dot{\xi}(s) \rangle ds.
\end{aligned}
$$

Hence, we proved the formula (12) for functions ξ of the form $\xi(s) = \xi_0\varphi(s)$, $s \in [0, T]$. Because such functions form a dense subspace in the space $C^1([0, T]; D(A^*))$, the lemma is true. ∎

3 Properties in general case

In this section we consider weak and mild solutions to the equation (1).

First we study the stochastic convolution defined by (8), that is,

$$W^\Psi(t) := \int_0^t S(t - \tau)\Psi(\tau)\, dW(\tau), \quad t \in [0, T].$$

Proposition 3 *Assume that $S(t)$, $t \geq 0$, are (as earlier) the resolvent operators corresponding to the Volterra equation (2). Then, for arbitrary process $\Psi \in \mathcal{N}^2(0, T; L_2^0)$, the process $W^\Psi(t)$, $t \geq 0$, given by (8) has a predictable version.*

Proof: Because proof of Proposition 3 is analogous to some schemes in theory of stochastic integral (see,e.g., [12, Chapter 4]) we provide only an outline of proof.

First, let us notice that the process $S(t-\tau)\Psi(\tau)$, where $\tau \in [0, t]$, belongs to $\mathcal{N}^2(0, T; L_2^0)$, because $\Psi \in \mathcal{N}^2(0, T; L_2^0)$. Then we may use the apparently well-known estimate (see, e.g., Proposition 4.16 in [7]): for arbitrary $a > 0$, $b > 0$ and $t \in [0, T]$

$$P(|W^\Psi(t)|_H > a) \le \frac{b}{a^2} + P\left(\int_0^t |S(t-\tau)\Psi(\tau)|_{L_2^0}^2 d\tau > b\right) . \qquad (15)$$

Because the resolvent operators $S(t)$, $t \ge 0$, are uniformly bounded on compact itervals (see [13]), there exists a constant $C > 0$ such that $||S(t)|| \le C$ for $t \in [0, T]$. So, we have $|S(t-\tau)\Psi(\tau)|_{L_2^0}^2 \le C^2 |\Psi(\tau)|_{L_2^0}^2$, $\tau \in [0, T]$.

Then the estimate (15) may be rewritten as

$$P(|W^\Psi(t)|_H > a) \le \frac{b}{a^2} + P\left(\int_0^t |\Psi(\tau)|_{L_2^0}^2 d\tau > \frac{b}{C^2}\right) . \qquad (16)$$

Let us consider predictability of the process W^Ψ in two steps. In the first step we assume that Ψ is an elementary process understood in the sense given in section 4.2 in [7]. In this case the process W^Ψ has a predictable version by Lemma 1, part (i).

In the second step Ψ is an arbitrary process belonging to $\mathcal{N}^2(0, T; L_2^0)$. Since elementary processes form a dense set in the space $\mathcal{N}^2(0, T; L_2^0)$, there exists a sequence (Ψ_n) of elementary processes such that for arbitrary $c > 0$

$$P\left(\int_0^T |\Psi(\tau) - \Psi_n(\tau)|_{L_2^0}^2 d\tau > c\right) \xrightarrow{n \to +\infty} 0 . \qquad (17)$$

By the previous part of the proof the sequence W_n^Ψ of convolutions

$$W_n^\Psi(t) := \int_0^t S(t-\tau)\Psi_n(\tau) dW(\tau)$$

converges in probability. Hence, it has a subsequence converging almost surely. This implies the predictability of the convolution $W^\Psi(t)$, $t \in [0, T]$.   ∎

Proposition 4 *Assume that $\Psi \in \mathcal{N}^2(0, T; L_2^0)$. Then the process $W^\Psi(t)$, $t \ge 0$, defined by (8) has square integrable trajectories.*

Proof: We have to prove that $\mathbb{E} \int_0^T |W^\Psi(t)|_H^2 dt < +\infty$. From Fubini's theo-

rem and properties of stochastic integral

$$\mathbb{E}\int_0^T \left|\int_0^t S(t-\tau)\Psi(\tau)dW(\tau)\right|_H^2 dt = \int_0^T \left[\mathbb{E}\left|\int_0^t S(t-\tau)\Psi(\tau)dW(\tau)\right|_H^2\right]dt$$

$$= \int_0^T\int_0^t |S(t-\tau)\Psi(\tau)|_{L_2^0}^2\,d\tau dt \leq M\int_0^T\int_0^t |\Psi(\tau)|_{L_2^0}^2\,d\tau dt < +\infty.$$

(from boundness of operators $S(t)$ and since $\Psi(\tau)$ are Hilbert-Schmidt)

∎

In the below result, the notions "parabolic" and "3-monotone" are understood in the sense defined by Prüss [13, Section 3].

Proposition 5 *Assume that (1) is parabolic, (VA) are satisfied and the kernel function a is 3-monotone. Let X be a predictable process with integrable trajectories. Assume that X has a version such that $P(X(t) \in D(A)) = 1$ for almost all $t \in [0,T]$ and (6) holds. If for any $t \in [0,T]$ and $\xi \in D(A^*)$*

$$\langle X(t), \xi\rangle_H = \langle X(0), \xi\rangle_H + \int_0^t \langle a(t-\tau)X(\tau), A^*\xi\rangle_H d\tau \tag{18}$$

$$+ \int_0^t \langle \xi, \Psi(\tau)dW(\tau)\rangle_H, \quad P-a.s.,$$

then

$$X(t) = S(t)X(0) + \int_0^t S(t-\tau)\Psi(\tau)dW(\tau), \quad t \in [0,T]. \tag{19}$$

Proof: For simplicity we omit index $_H$ in the inner product. First, we see, analogously like in Lemma 2, that if (18) is satisfied, then

$$\langle X(t), \xi(t)\rangle = \langle X(0), \xi(0)\rangle + \int_0^t \langle (\dot{a}\star X)(\tau) + a(0)X(\tau), A^*\xi(\tau)\rangle d\tau$$

$$+ \int_0^t \langle \Psi(\tau)dW(\tau), \xi(\tau)\rangle + \int_0^t \langle X(\tau), \dot{\xi}(\tau)\rangle d\tau, \quad P-a.s. \tag{20}$$

holds for any $\xi \in C^1([0,t], D(A^*))$ and $t \in [0,T]$.

Because (1) is parabolic and a is 2-regular (what is implied by 3-monotone), then, by [13, Theorem 3.1], there exists a resolvent $S \in C^1((0,+\infty); L(H))$ for (1).

Now, let us take $\xi(\tau) := S^*(t-\tau)\zeta$ with $\zeta \in D(A^*)$, $\tau \in [0,t]$. The equation (20) may be written like

$$\langle X(t), S^*(0)\zeta\rangle = \langle X(0), S^*(t)\zeta\rangle + \int_0^t \langle (\dot{a}\star X)(\tau) + a(0)X(\tau), A^*S^*(t-\tau)\zeta\rangle d\tau$$

$$+ \int_0^t \langle \Psi(\tau)dW(\tau), S^*(t-\tau)\zeta\rangle + \int_0^t \langle X(\tau), (S^*(t-\tau)\zeta)'\rangle d\tau,$$

where derivative ()' in the last term is taken over τ.

Next, using $S^*(0) = I$, we rewrite

$$\langle X(t), \zeta \rangle = \langle S(t)X(0), \zeta \rangle + \int_0^t \langle S(t-\tau)A \left[\int_0^\tau \dot{a}(\tau-\sigma)X(\sigma)d\sigma + a(0)X(\tau) \right], \zeta \rangle d\tau$$

$$+ \int_0^t \langle S(t-\tau)\Psi(\tau)dW(\tau), \zeta \rangle + \int_0^t \langle \dot{S}(t-\tau)X(\tau), \zeta \rangle d\tau. \qquad (21)$$

To prove (19) it is enough to show that the sum of the first integral and the third one in the equation (21) gives zero.

We use properties of resolvent operators and the derivative $\dot{S}(t-\tau)$ with respect to τ. Then

$$I := \left\langle \int_0^t \dot{S}(t-\tau)X(\tau)d\tau, \zeta \right\rangle = \left\langle - \int_0^t \dot{S}(\tau)X(t-\tau)d\tau, \zeta \right\rangle$$

$$= \left\langle - \left(\int_0^t \left[\int_0^\tau \dot{a}(\tau-s)AS(s)ds \right] X(t-\tau)d\tau - \int_0^t a(0)AS(\tau)X(t-\tau)d\tau \right), \zeta \right\rangle$$

$$= \langle -([A(\dot{a} \star S)(\tau) \star X](t) + a(0)A(S \star X)(t)), \zeta \rangle.$$

The kernel function a is 3-monotone, so $a \in C^1(\mathbb{R}_+)$, and then has bounded variation. Hence, the convolution $(a \star S)(\tau)$ has sense (see [13, Section 1.6] or [1]).

Since

$$\int_0^t \langle a(0)AS(t-\tau)X(\tau), \zeta \rangle d\tau = \int_0^t \langle a(0)AS(\tau)X(t-\tau), \zeta \rangle d\tau$$

and

$$J := \int_0^t \langle S(t-\tau)A \left[\int_0^\tau \dot{a}(\tau-\sigma)X(\sigma)d\sigma \right], \zeta \rangle d\tau = \int_0^t \langle AS(t-\tau)(\dot{a} \star X)(\tau), \zeta \rangle d\tau$$

$$= \langle A(S \star (\dot{a} \star X)(\tau))(t), \zeta \rangle = \langle A((S \star \dot{a})(\tau) \star X)(t), \zeta \rangle \text{ for any } \zeta \in D(A^*),$$

so $J = -I$, hence $J + I = 0$. This means that (19) holds for any $\zeta \in D(A^*)$. Since $D(A^*)$ is dense in H^*, then (19) holds. ∎

Remark: In Proposition 5, the assumption that a is 3-monotone may be replaced by both: 2-regularity of a and $a \in BV_{\text{loc}}(\mathbb{R}_+)$.

Comment: Proposition 5 shows that under particular conditions a weak solution to (1) is a mild solution to the equation (1).

Proposition 6 *Let* VOLTERRA ASSUMPTIONS *be satisfied. If the process* $\Psi \in \mathcal{N}^2(0, T; L_2^0)$, *then the stochastic convolution* W^Ψ *fulfills the equation (18).*

Proof: Let us notice that the process W^Ψ has integrable trajectories. For any $\xi \in D(A^*)$ we have

$$\int_0^t \langle a(t-\tau)W^\Psi(\tau), A^*\xi\rangle_H d\tau \equiv \text{(from (8))}$$

$$\equiv \int_0^t \langle a(t-\tau)\int_0^\tau S(\tau-\sigma)\Psi(\sigma)dW(\sigma), A^*\xi\rangle_H d\tau =$$

(from Dirichlet's formula and stochastic Fubini's theorem)

$$= \int_0^t \langle \left[\int_\sigma^t a(t-\tau)S(\tau-\sigma)d\tau\right]\Psi(\sigma)dW(\sigma), A^*\xi\rangle_H$$

$$= \int_0^t \langle A\left[\int_0^{t-\sigma} a(t-\sigma-z)S(z)dz\right]\Psi(\sigma)dW(\sigma), \xi\rangle_H$$

(where $z := \tau - \sigma$ and from definition of convolution)

$$= \langle \int_0^t A[(a \star S)(t-\sigma)]\Psi(\sigma)dW(\sigma), \xi\rangle_H =$$

(from the equation (4), because $A(a \star S)(t-\sigma)x = (S(t-\sigma) - I)x$, where $x \in H$)

$$= \langle \int_0^t [S(t-\sigma) - I]\Psi(\sigma)dW(\sigma), \xi\rangle_H =$$

$$= \langle \int_0^t S(t-\sigma)\Psi(\sigma)dW(\sigma), \xi\rangle_H - \langle \int_0^t \Psi(\sigma)dW(\sigma), \xi\rangle_H.$$

Hence, we obtained the following equation

$$\langle W^\Psi(t), \xi\rangle_H = \int_0^t \langle a(t-\tau)W^\Psi(\tau), A^*\xi\rangle_H d\tau + \int_0^t \langle \xi, \Psi(\tau)dW(\tau)\rangle_H$$

for any $\xi \in D(A^*)$. ∎

Corollary 1 *Let* VOLTERRA ASSUMPTIONS *hold with a bounded operator A. If Ψ belongs to $\mathcal{N}^2(0, T; L_2^0)$ then*

$$W^\Psi(t) = \int_0^t a(t-\tau)AW^\Psi(\tau)d\tau + \int_0^t \Psi(\tau)dW(\tau). \tag{22}$$

Comment: The formula (22) says that the convolution W^Ψ is a strong solution to (1) if the operator A is bounded.

The below theorem is a consequence of the results obtained up to now.

Theorem 2 *Suppose that (VA) and (PA) hold. Then a strong solution (if exists) is always a weak solution of (1). If, additionally, assumptions of Proposition 5 are satisfied, a weak solution is a mild solution to the Volterra equation (1). Conversely, under conditions of Proposition 6, a mild solution X is also a weak solution to (1).*

4   Some estimates

In this section we provide two estimates for stochastic convolution (8). Some considerations (see, e.g. [11], where maximal type inequalities for the equation (1) were studied) show that in general case, that is when (VA) are supposed only, it is very difficult to say something interesting about regularity of the convolution (8). Similar situation was in the semigroup case, see [7], where regularity results have been received under additional assumptions on semigroups, for instance when contractions or analytical semigroups were studied.

As we have already written, Clément and Da Prato (see, [3] and [4]) obtained some regularity results in the case when A was a self-adjoint operator satisfying some technical assumptions, the function a was completely positive and when $\Psi(t) \equiv I$, that is for the convolution $\widetilde{W}(t) := \int_0^t S(t-\tau)dW(\tau)$. In their case, $||S(t)|| \leq 1$, what is a resolvent analogon of contraction semigroup.

Theorem 3 *If* $\Psi \in \mathcal{N}^2(0,T;L_2^0)$ *then the following estimate holds*

$$\sup_{t \leq T} \mathbb{E}(|W^\Psi(t)|_H) \leq C\, M_T\, \mathbb{E} \left(\int_0^T |\Psi(t)|_{L_2^0}^2 \, dt \right)^{\frac{1}{2}}, \tag{23}$$

where C is a constant and $M_T = \sup_{t \leq T} ||S(t)||$.

Comment: The estimate (23) seems to be rather coarse. It comes directly from the definition of stochastic integral. Since (23) reducest to the Davis inequality for msrtingales if $S(\cdot) = I$, the constant C appeared on the right hand side. Unfortunately, we can not use more refined tools, for instance Itô's formula (see, e.g. [14] for Tubaro's estimate), because the process W^Ψ is not enough regular.

The next theorem is a consequence of Theorem 3.

Theorem 4 *Assume that* $\Psi \in \mathcal{N}^2(0,T;L_2^0)$. *Then*

$$\sup_{t \leq T} \mathbb{E}(|W^\Psi(t)|_H) \leq \widetilde{C}(T)\, |\Psi|_{\mathcal{N}^2(0,T;L_2^0)},$$

where a constant $\widetilde{C}(T)$ depends on T.

Proof: From (8) and property of stochastic integral we have

$$
\begin{aligned}
\mathbb{E}(|W^\Psi(t)|_H) &= \mathbb{E} \left(|\int_0^t S(t-\tau)\Psi(\tau)dW(\tau)|_H \right) \\
&\leq C\, \mathbb{E} \left(\int_0^t |S(t-\tau)\Psi(\tau)|_{L_2^0}^2 \, d\tau \right)^{\frac{1}{2}}
\end{aligned}
$$

(from writing out the Hilbert-Schmidt norm)

$$\leq \quad C \, \mathbb{E} \left(\int_0^t ||S(t-\tau)||^2 \, |\Psi(\tau)|^2_{L_2^0} \, d\tau \right)^{\frac{1}{2}}$$

$$\leq \quad C \, M_T \, \mathbb{E} \left(\int_0^t |\Psi(\tau)|^2_{L_2^0} \, d\tau \right)^{\frac{1}{2}} \leq \text{ (by the Hölder inequality)}$$

$$\leq \quad C \, M_T \, \left(\mathbb{E} \int_0^t |\Psi(\tau)|^2_{L_2^0} \, d\tau \right)^{\frac{1}{2}} \leq \widetilde{C}(T) \, |\Psi|_{\mathcal{N}^2(0,T;L_2^0)} \ ,$$

where M_T is as above and $\widetilde{C}(T) = C \, M_T$. ∎

Summing up the paper, it is worth to emphasize that better and more sophisticated regularity results should be obtained for exponentially bounded and analytical resolvents. The situation is similar to that for semigroup case, when the best results are reached for analytical semigroups.

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