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Characterizations of Algebraic L-domains and Equivalence of Relevant Categories[1]

Xu-Xin Mao

College of Science
Nanjing University of Aeronautics and Astronautics
Nanjing 210016, P. R. China

Luo-Shan Xu[2]

Department of Mathematics, Yangzhou University
Yangzhou Jiangsu 225002, P. R. China

Abstract

This paper concerns algebraic L-domains. Main results are: (1) All the compact elements of an algebraic L-domain forms an L-cusl; (2) An algebraic L-domain equipped with the Scott topology is the sobrification of the set of all compact elements equipped with the Alexandrov topology; (3) The ideal completion of an L-cusl is an algebraic L-domain. (4) It is also proved that with proper concepts of L-cusl embeddings and projection pairs, the category of algebraic L-domains and projection pairs is equivalent to the category of L-cusls and L-cusl embeddings.

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1 Introduction

Domain theory plays a fundamental role in denotational semantics of programming languages. Its importance may also be judged from the fact that it has had many applications in fields as diverse as general topology, lattice theory,

[1]Supported by the NSF of China (10371106) and by Fund (S0667-082) from NUAA.
[2]Corresponding author. E-mail: luoshanxu@hotmail.com, luoshanxu@yahoo.com

category theory and theoretical computer science as well as in many other areas of mathematics (see, for example, [1, 2, 6, 11, 12]).

L-domains, a particular class of domains, were firstly introduced by Jung in [4] and have received more and more attention because of their good properties (see, for example, [2, 4], [7]-[10]). This paper will focus on algebraic L-domains and relevant categories by exploring characterizations for them.

For the sake of clarity, we recall some basic notions (see, e.g., [1, 2]) as preliminaries.

Let P be a poset. A subset A of P is said to be *consistent* if A has an upper bound in P. The set of all ideals of P ordered by set inclusion is called the *ideal completion* of P, denoted by $Idl(P)$. A poset P with least element is called a *conditional upper semilattice* (in short, *cusl*) (see [12]) if $a \vee b$ exists whenever $\{a, b\} \subseteq P$ is consistent. A poset in which every directed set has the supremum is called a *dcpo*.

Let P be a dcpo, $x, y \in P$. We say that x *approximates* y, written $x \ll y$, if whenever D is directed with $\sup D \geq y$, then $x \leq d$ for some $d \in D$. An element $k \in P$ is said to be compact, if $k \ll k$. The set of all compact elements of P is denoted by $K(P)$. If for every element $x \in P$, the set $\twoheaddownarrow x := \{a \in P : a \ll x\}$ is directed and $\sup \twoheaddownarrow x = x$, then P is said to be *continuous*. If every element in P is a directed sup of compact elements, then P is said to be *algebraic*. A continuous dcpo is called a *domain*. A domain P is called an *L-domain* if $\forall x \in P$, $\downarrow x$ is a complete lattice. An upper set U of P is said to be Scott open if for all directed sets $D \subseteq P$, $\vee^{\uparrow} D \in U$ implies $U \cap D \neq \emptyset$. All the Scott open sets of P forms a topology, called the *Scott topology*, denoted $\sigma(P)$. If a map $f : P \to Q$ is continuous from topological spaces $(P, \sigma(P))$ to $(Q, \sigma(Q))$, then f is said to be *Scott continuous*. It is well known that f is Scott continuous iff f preserves all directed suprema. A *Galois connection* (see [2]) between two posets S and T is a pair (g, d) of order preserving maps $g : S \to T$ and $d : T \to S$ such that $gd \geq id_T$ and $dg \leq id_S$.

2 Algebraic Characterization Theorem

We first generalize the concept of cusls to the concept of L-cusls.

Definition 2.1. Let P be a poset. If for all $p \in P$, the principal ideal $\downarrow p$ is a cusl, then P is called a locally conditional upper semilattice (in short, L-cusl).

Lemma 2.2. *Let P be a poset and $A \subseteq P$. Let s, t be two upper bounds of A. If A has a supremum in $\downarrow s$, denoted by $\vee_s A$ and has a supremum in $\downarrow t$, denoted by $\vee_t A$, then $\vee_s A = \vee_t A$ whenever $\vee_s A \leq t$ or $\vee_t A \leq s$.*

Proof. Suppose that $\vee_s A \leq t$ without losing generality. Then $\vee_s A$ is an upper bound of A in $\downarrow t$ and $\vee_t A \leq \vee_s A \leq s$. This reveals that $\vee_t A$ is also an upper bound of A in $\downarrow s$. So $\vee_s A \leq \vee_t A$ and $\vee_s A = \vee_t A$. □

Lemma 2.3. *Let L be an L-domain, $a \in L$. If $x, y \in \downarrow_{K(L)} a$, then $x \vee_a y \in \downarrow_{K(L)} a$, where $\downarrow_{K(P)} a = \{k \in K(P) : k \leq a\}$ and $x \vee_a y$ is the join of x and y in $\downarrow a$.*

Proof. Suppose that $D \subseteq L$ is a directed set of L and $x \vee_a y \leq t = \vee^{\uparrow} D$. Then $x \vee_a y \in \downarrow t$. Since L is an L-domain, the join of x and y in $\downarrow t$ exists, denoted by $x \vee_t y$. By Lemma 2.2, $x \vee_a y = x \vee_t y$. By the compactness of x, y and $x \vee_a y \leq t = \vee^{\uparrow} D$, there exist d_1, $d_2 \in D$ such that $x \leq d_1$ and $y \leq d_2$. By the directedness of D, there is some $d_3 \in D$ such that x, $y \leq d_3 \leq t$. Thus $x \vee_t y = x \vee_a y \leq d_3$. This shows that $x \vee_a y$ is a compact element and $x \vee_a y \in \downarrow_{K(L)} a$. □

Proposition 2.4. *If L is an algebraic L-domain, then $K(L)$ is an L-cusl.*

Proof. This follows from Definition 2.1 and Lemma 2.3. □

Proposition 2.5. *If P is an L-cusl, then $(Idl(P), \subseteq)$ is an algebraic L-domain. Furthermore, $P \cong K(Idl(P)) = \{\downarrow a : a \in P\}$.*

Proof. This is straightforward with the above two lemmas. □

Proposition 2.6. *If L is an algebraic L-domain. Then $L \cong Idl(K(L))$.*

Proof. Define $f : Idl(K(L)) \to L$ such that $f(I) = \sup I$ for all $I \in Idl(K(L))$. Define $g : L \to Idl(K(L))$ such that $g(x) = \twoheaddownarrow x \cap K(L)$ for all $x \in L$. It is easy to show that f and g give mutual inverse isomorphisms for $L \cong Idl(K(L))$. □

Theorem 2.7. (The Algebraic Characterization Theorem) *Let L be a dcpo. Then L is an algebraic L-domain if and only if there is some L-cusl P such that $L \cong Idl(P)$.*

Proof. "$\Rightarrow$": Take $P = (K(L), \leq)$. Then by Proposition 2.4 and Proposition 2.6, P is an L-cusl and $L \cong Idl(P)$.

"$\Leftarrow$": By Proposition 2.5. □

Theorem 2.8. *Let P be an L-cusl and $Idl(P)$ its ideal completion. Let $r : P \to Idl(P)$ be the canonical embedding given by $r(p) = \downarrow p$ for all $p \in P$. Suppose L is an algebraic L-domain and $g : P \to L$ is monotone. Then there exists a unique continuous map $\tilde{g} : Idl(P) \to L$ such that $\tilde{g}r = g$.*

Proof. Define $\tilde{g} : Idl(P) \to L$ such that $\forall I \in Idl(P)$, $\tilde{g}(I) = \vee^{\uparrow}\{g(a) : a \in I\}$. Then g is what we need. □

Let **ALDom** be the category of algebraic L-domains and Scott continuous maps. Let **LCusl** be the category of L-cusls and monotone maps. By Theorem 2.8, We have

Theorem 2.9. **ALDom** *is a reflective subcategory of category* **LCusl**.

3 Topological Characterization Theorem

To extend our view, this section gives a simple topological characterization of algebraic L-domains by the technique of round ideal completions and sobrifications.

Recall that (see [1, 5]) a binary relation $\prec$ on a set P is called fully transitive if it is transitive ($x \prec y$, $y \prec z \Rightarrow x \prec z$) and satisfies the interpolation property:

$$\forall |F| < \infty,\ F \prec z \Rightarrow \exists y \prec z \text{ such that } F \prec y,$$

where $F \prec y$ means $\forall t \in F$, $t \prec y$. An abstract basis $(B, \prec)$ is a set equipped with a binary relation which is fully transitive.

Definition 3.1. ([1, 5]) Let $(B, \prec)$ be an abstract basis. A non-empty subset I of B is a round ideal if

(1) $\forall y \in I$, $x \prec y \Rightarrow x \in I$;

(2) $\forall x, y \in I$, $\exists z \in I$ such that $x \prec z$ and $y \prec z$.

The set of all round ideals of B ordered by set inclusion is called the round ideal completion of B, denoted by $RI(B)$.

Definition 3.2. ([5]) Let $(B, \prec)$ be an abstract basis. For $b \in B$, the set $\{x \in B : b \prec x\}$ is denoted by $\Uparrow b$. Then the set $\{\Uparrow b : b \in B\}$ forms a basis for a topology on B, called the pseudoScott topology.

Definition 3.3. ([5]) Let $(X, \Omega(X))$ be a topological space. A pair (X^s, j) is called a sobrification of X if X^s is a sober space and $j : X \to X^s$ is a continuous map such that $j^{-1} : \Omega(X^s) \to \Omega(X)$ is a lattice isomorphism.

Lemma 3.4. ([5]) *Let $(X, \Omega(X))$ be a topological space. If (X_i^s, j_i) are sobrifications of X for i=1, 2, then there exists a unique homeomorphism $h : X_1^s \to X_2^s$ such that $h \circ j_1 = j_2$.*

Lemma 3.5. ([5]) *Let $(B, \prec)$ be an abstract basis equipped with the pseudoScott topology. Then $(RI(B), j)$ is a sobrification of B, where $RI(B)$ is equipped with the Scott topology and $\forall b \in B$, $j(b) = \Downarrow b = \{x \in B : x \prec b\}$.*

Remark 3.6. Let L be an algebraic (L-)domain. Let $\Upsilon(K(L))$ be the Alexandrov topology formed by all the upper sets of $K(L)$. Then $(K(L), \leq)$ is a(n abstract) basis of L, and by Proposition 2.6, $L \cong Idl(K(L)) = RI(K(L))$. In this case the pseudoScott topology on $K(L)$ is exactly the Alexandrov topology.

Theorem 3.7. (The Topological Characterization Theorem) *Let L be an L-domain. Then L is an algebraic L-domain if and only if the pair (L, j) is a sobrification of $(K(L), \Upsilon(K(L)))$, where L is equipped with the Scott topology and $j : K(L) \to L$ is the map defined for all $k \in K(L)$, $j(k) = k$.*

Proof. "⇒": This follows from Lemma 3.5 and Remark 3.6.
"⇐": Clearly $(K(L), \leq)$ is an abstract basis. So by Lemma 3.5 and Remark 3.6, $(Idl(K(L)), j^*)$ is a sobrification of $(K(L), \Upsilon(K(L)))$, where $Idl(K(L))$ is equipped with the Scott topology and $\forall k \in K(L)$, $j^*(k) = \downarrow k \cap K(L)$. Since (L, j) is also a sobrification of $(K(L), \Upsilon(K(L)))$, by Lemma 3.4, there is a unique homeomorphism $h : L \to Idl(K(L))$ such that $h \circ j = j^*$. This implies that h and h^{-1} are both Scott continuous and thus monotone, showing that h is an order isomorphism for $L \cong Idl(K(L))$. It is well known that $Idl(K(L))$ is algebraic, hence, L is an algebraic L-domain. □

4 L-cusl Embeddings and Projection Pairs

To consider the categorical aspect of algebraic L-domains, this section technically introduce some relevant morphisms.

Definition 4.1. Let $f : P \to Q$ be a map, where P and Q are L-cusls. Then f is called an L-cusl embedding if the following conditions hold.

(1) $a \leq b \Leftrightarrow f(a) \leq f(b)$;

(2) $\forall p \in P$, $\forall a, b \in \downarrow p$, $f(a \vee_p b) = f(a) \vee_{f(p)} f(b)$;

(3) For all finite $A \subseteq P$, if $f(A)$ has an upper bound t in Q, then there exists some $s \in P$ such that s is an upper bound of A in P and $f(s) \leq t$.

Proposition 4.2. *Let P, Q and R be L-cusls. Then*

(i) The identity map $id : P \to P$ is an L-cusl embedding;

(ii) If $f : P \to Q$ and $g : Q \to R$ are L-cusl embeddings, then $gf : P \to R$ is also an L-cusl embedding.

Proof. Straightforward. □

In view of this proposition, all the L-cusls and L-cusl embeddings forms a category, denoted by $\mathbf{LCusl^e}$.

Proposition 4.3. *If $f : P \to Q$ is an L-cusl embedding, then $\forall I \in IdlQ$, $f^{-1}(I) \in IdlP$.*

Proof. Straightforward. □

Example 4.4. We give an example to show the converse of the above proposition is not true. Let $L = \{0, a, b, 1\}$ with $0 < a$, $b < 1$ and a, b being incomparable. Let $M = \{0, a, 1\}$ with $0 < a < 1$. Then L and M are all lattices. Define $f : L \to M$ such that $f(0) = 0, f(a) = a, f(b) = f(1) = 1$. Then the inverse image of an ideal of M under f is an ideal. But, clearly, f is not an L-cusl embedding because of Definition 4.1(1).

Definition 4.5. Let L and M be dcpos and let $f : L \to M$ and $f' : M \to L$ be continuous maps. If $ff' \leq id_M$ and $f'f = id_L$, then (f, f') is called a projection pair for (L, M). In this case we call f an embedding and f' the corresponding projection.

Proposition 4.6. *Let (f, f') be a projection pair for (L, M), where L and M are continuous dcpos. Then f' is uniquely determined by the embedding f.*

Proof. By the above definition and [2, Theorem 0-3.6], it is easy to see that (f', f) is a special Galois connection. And the proposition is clear by [2, Theorem 0-3.2]. □

Proposition 4.7. *Let L and M be dcpos and (f, f') a projection pair for (L, M). Then*

(i) If (g, g') is another projection pair for (L, M), then $f \leq g \Leftrightarrow f' \geq g'$.

(ii) $\forall x, y \in L$, $x \leq y \Leftrightarrow f(x) \leq f(y)$.

(iii) $\forall x \in L$, $\forall y \in M$, $f(x) \leq y \Leftrightarrow x \leq f'(y)$.

(iv) $f(K(L)) \subseteq K(M)$.

Proof. Straightforward. □

Proposition 4.8. *Let L, M and Q be dcpos. Then*

(i) (id_L, id_L) is a projection pair for (L, L).

(ii) If (f, f') and (g, g') are projection pairs for (L, M) and (M, Q) respectively, then $(gf, f'g')$ is a projection pair for (L, Q).

Proof. Straightforward. □

In view of Proposition 4.8, all the dcpos and projection pairs forms a category, denoted by $\mathbf{DCPO^{ep}}$, where the composition of two projection pairs (f, f') for (L, M) and (g, g') for (M, Q) is the projection pair $(gf, f'g')$ for (L, Q) as in Proposition 4.8. All the algebraic L-domains forms a full subcategory of $\mathbf{DCPO^{ep}}$, denoted by $\mathbf{ALDom^{ep}}$.

Theorem 4.9. *Let L and M be algebraic L-domains and $f : L \to M$ a continuous map. Then f is an embedding if and only if $f(K(L)) \subseteq K(M)$ and $f|_{K(L)} : K(L) \to K(M)$ is an L-cusl embedding.*

Proof. "$\Rightarrow$": Suppose that $f : L \to M$ is an embedding and $f' : M \to L$ is the corresponding projection. Then $f(K(L)) \subseteq K(M)$ follows from Proposition 4.7 (iv).

It is straightforward to show that $f|_{K(L)} : K(L) \to K(M)$ is an L-cusl embedding, i.e., to verify conditions (1)-(3) of Definition 4.1.

"$\Leftarrow$": Suppose that $f(K(L)) \subseteq K(M)$ and $f|_{K(L)} : K(L) \to K(M)$ is an L-cusl embedding. Let $B_y = \{a \in K(L) : f(a) \leq y\}$ for all $y \in M$. Take $t \in \downarrow y \cap K(M) \neq \emptyset$. Since $f|_{K(L)} : K(L) \to K(M)$ is an L-cusl embedding, there is some $s \in K(L)$ such that $f(s) \leq t \leq y$ and $s \in B_y \neq \emptyset$. It is easy to show that B_y is directed. Define $g : M \to L$ such that $g(y) = \vee^{\uparrow} B_y$ for all $y \in M$. Then g is well-defined and is continuous.

Next we show that (f, g) is a projection pair for (L, M). For all $y \in M$, we have

$$fg(y) = f(\vee^{\uparrow} B_y) = \vee^{\uparrow}\{f(a) : a \in B_y\} \leq \vee^{\uparrow}(\downarrow y \cap K(M)) = y,$$

i.e., $fg \leq id_M$. Since $B_{f(x)} = \{a \in K(L) : f(a) \leq f(x)\} \supseteq \downarrow x \cap K(L)$ for all $x \in L$, we have $gf(x) = \vee^{\uparrow} B_{f(x)} \geq \vee^{\uparrow}(\downarrow x \cap K(L)) = x$. On the other hand, for all $a \in \downarrow x \cap K(L)$, we have $gf(a) = \vee^{\uparrow} B_{f(a)} = a \leq x$ and $gf(x) \leq x$ by $\vee^{\uparrow}(\downarrow x \cap K(L)) = x$. Hence $gf(x) = x$, i.e., $gf = id_L$. To sum up, (f, g) is a projection pair for (L, M). □

Corollary 4.10. *Let P be an L-cusl and L an algebraic L-domain. If $f : P \to L$ is an L-cusl embedding and $f(P) \subseteq K(L)$, then there exists a unique embedding $\tilde{f} : Idl(P) \to L$ such that $\tilde{f}r = f$, where $r : P \to Idl(P)$ such that $\forall p \in P, r(p) = \downarrow p$.*

Proof. This follows from Proposition 2.5, 2.6, Theorem 2.8 and 4.9. □

Theorem 4.11. *Let P, Q be L-cusls and $Idl(P)$, $Idl(Q)$ be their ideal completions. Then there exists an L-cusl embedding from P into Q if and only if there exists a projection pair for $(Idl(P), Idl(Q))$.*

Proof. "$\Leftarrow$": Suppose that (f, f') is a projection pair for $(Idl(P), Idl(Q))$. Let $g = f|_{K((Idl(P))} : K(Idl(P)) \to K(Idl(Q))$. Then g is an L-cusl embedding by Theorem 4.9. Since $K(Idl(P)) \cong P$ and $K(Idl(Q)) \cong Q$, one can easily construct with g an L-cusl embedding from P into Q.

"$\Rightarrow$": Suppose that $g : P \to Q$ is an L-cusl embedding. Define $f : P \to Idl(Q)$ for all $p \in P$, $f(p) = \downarrow g(p)$. By Proposition 2.5, we have $f(P) \subseteq K(Idl(Q)) = \{\downarrow a : a \in Q\}$. Since g is an L-cusl embedding, it is straightforward to show that f is also an L-cusl embedding. By Corollary 4.10, there exists an embedding $\tilde{f}$ from $Idl(P)$ to $Idl(Q)$. This shows that there exists a projection pair for $(Idl(P), Idl(Q))$. □

5 Equivalence Theorem of Categories

In this section, we consider the equivalence of categories $\mathbf{LCusl^e}$ and $\mathbf{ALDom^{ep}}$. Some basic concepts of category theory please refer to [3, 13].

Proposition 5.1. *For an L-cusl embedding $f : P \to Q$, where P and Q are L-cusls, define $f^* : Idl(P) \to Idl(Q)$ such that $\forall I \in Idl(P)$, $f^*(I) = \{b \in Q : \exists a \in I$ such that $b \leq f(a)\}$. Then f^* is an embedding.*

Proof. It is not difficult with Theorem 4.9 to prove that f^* is an embedding. □

Proposition 5.2. *Define $Idl : \mathbf{Lcusl^e} \to \mathbf{ALDom^{ep}}$ as follows.*

(i) For all $P \in ob(\mathbf{Lcusl^e})$, $Idl(P)$ is the ideal completion of P;

(ii) For an L-cusl embedding $f : P \to Q$, where P and Q are L-cusls, define $Idl(f) = (f^*, f^{*\prime})$, where $f^* : Idl(P) \to Idl(Q)$ is defined as in Proposition 5.1 and $f^{*\prime} : Idl(Q) \to Idl(P)$ is the corresponding projection determined by f^*. Then $Idl : \mathbf{Lcusl^e} \to \mathbf{ALDom^{ep}}$ is a functor.

Proof. It is easy to see by Proposition 4.6 and Proposition 5.1 that $Idl(f)$ is well-defined. Then under the composition of two projection pairs defined immediately after Proposition 4.8, it is straightforward to show that Idl is a functor. □

Proposition 5.3. *Define $K : \mathbf{ALDom^{ep}} \to \mathbf{Lcusl^e}$ as follows.*

(i) For $L \in ob(\mathbf{ALDom^{ep}})$, define $K(L)$ to be the set of compact elements of L;

(ii) For any $(f, f') \in Mor(\mathbf{ALDom^{ep}})$, a projection pair for (L, M), where L and M are algebraic L-domains, define $K(f, f') = f|_{K(L)}$. Then K is a functor.

Proof. It follows from Theorem 4.9 that $f|_{K(L)}$ is an L-cusl embedding. So, $K(f, f')$ is well-defined. Then, it is straightforward to show that K is a functor. □

Theorem 5.4. *Categories $\mathbf{Lcusl^e}$ and $\mathbf{ALDom^{ep}}$ are equivalent.*

Proof. Define $\alpha : id_{\mathbf{Lcusl^e}} \to K \circ Idl$ such that $\alpha(P) = r_P : P \to K(Idl(P))$ for all $P \in ob(\mathbf{Lcusl^e})$, where for all $a \in P$, $r_P(a) = \downarrow a$. Then by Proposition 2.5, r_P is an isomorphism. Next we show that α is a natural transformation, i.e., show for any L-cusl embedding $f : P \to Q$, $(K \circ Idl)(f) r_P = r_Q id(f)$. In fact, for all $a \in P$, we have

$$(K \circ Idl)(f) r_P(a) = (K \circ Idl)(f)(\downarrow a) = f^*(\downarrow a) = \downarrow f(a) = r_Q(f(a)) = r_Q id(f)(a).$$

So, α is indeed a natural transformation.

Define $j : id_{\mathbf{ALDom^{ep}}} \to Idl \circ K$ such that for all $L \in ob(\mathbf{ALDom^{ep}})$, $j(L) = (j_L, j'_L)$, where $j_L : L \to Idl(K(L))$ is defined for all $a \in P$, $j_L(a) =$

$\downarrow a \cap K(L)$ and j'_L is the corresponding projection of j_L. Clearly j_L is monotone and injective. Claim that j_L is also surjective. In fact, for every $I \in Idl(K(L))$, let $b = \sup_L I$. Then $j(b) = \downarrow b \cap K(L) \supseteq I$. If $x \in \downarrow b \cap K(L)$, then by the compactness of x there is some $c \in I$ such that $x \leq c$ and $x \in I$, showing that $\downarrow b \cap K(L) \subseteq I$ and $j(b) = I$. So, j_L is surjective and thus an isomorphism. By Proposition 4.6, $j'_L = j_L^{-1}$ and (j_L, j'_L) is an isomorphism in $\mathbf{ALDom^{ep}}$.

Next we show that $j : id_{\mathbf{ALDom^{ep}}} \to Idl \circ K$ is a natural transformation, or equivalently to show that for any projection pair (f, f') for (L, M), $(Idl \circ K)(f, f')j(L) = j(M)id(f, f')$. In fact,

$$(Idl \circ K)(f, f')j(L) = (Idl \circ K)(f, f')(j_L, j'_L) = ((f|_{K(L)})^* j_L, j'_L(f|_{K(L)})^{*\prime})$$

and $j(M)id(f, f') = (j_M f, f' j'_M)$. By Proposition 4.6, it suffices to show $(f|_{K(L)})^* j_L = j_M f$. For all $x \in L$, we have

$$(f|_{K(L)})^* j_L(x) = (f|K(L))^*(\downarrow x \cap K(L)) = \downarrow f(x) \cap K(M) = j_M f(x),$$

showing that $f|_{K(L)})^* j_L = j_M f$. Thus $((f|^*_{K(L)} j_L, j'_L(f|_{K(L)})^{*\prime}) = (j_M f, f' j'_M)$ and $(Idl \circ K)(f, f')j(L) = j(M)id(f, f')$. So j is a natural transformation.

To sum up, α and j are all natural isomorphisms and thus categories $\mathbf{Lcusl^e}$ and $\mathbf{ALDom^{ep}}$ are equivalent. □

Remark 5.5. In [8], Lawson and Xu introduce the concept of sL-domains, a generalization of L-domains. It should be noted that similar results about algebraic sL-domains can also be obtained by introducing the concepts of sL-cusls, a generalization of L-cusls, and sL-cusl embeddings as well as relevant categories. The details are left to the reader.

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