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Equi-affine Vector Fields on Manifold with Equi-affine Structure

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Abstract

The aim is to extend the theory of affine plane curves on $\mathbb{R}^2$, to the manifolds of 2 and n dimensions. Affine arclength , affine curvatures, and equi-affine vector fields are defined on a manifold of dimension n with equi-affine structure. Classification of equi-affine vector fields on two dimensional equi-affine structure is obtained.

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1 Introduction

In equi-affine differential geometry, the objects are invariant under the transformations, keeping the area (in $\mathbb{R}^2$) and the volume (in $\mathbb{R}^3$)fixed.

In differential geometry, when the parameter of a curve is arclength, then the acceleration and velocity vectors are perpendicular.The length of acceleration vector is called Euclidian curvature, which is equal to the area of a rectangle made by the acceleration and velocity vectors.

Area is defined in two dimensional equi-affine geometry, but arclength is not defined. In [4] and [8] , the notion of arclength is extended such that, if the area of parallelogram made by the velocity and acceleration vectors is unit, then the parameter of the curve is called affine arclength. If $x : I \to R^2$ has define by arclength parameter i.e $|x'\ x''| = 1$, then $x''' = -kx'$ and $k = |x''\ x'''|$ is called affine curvature, in other words the area of parallelogram made by x'' and x''' is the affine curvature.

In [2, 3, 4, 6] the notion of affine arclength is defined for 3 and n dimensions, such that parameter of the curve $x : I \to \mathbb{R}^n$ is parameter of affine arclength if along the curve x, $\{x'(t), \cdots, x^{(n)}(t)\}$ is an equi-affine base, i.e the volume of the parallelopiped made by base vectors is unit .

In section 2, we introduce equi-affine structures and some its properties. In section 3, we define equi-affine vector fields on equi-affine structure, and we extend the concept of affine arc-length parameter to equi-affine structure.In section 4, we define affine curvature of curves on 2-dimensional equi-affine manifold and deduce some of its properties. In section 5,these notions are extended to n-dimensional equi-affine manifolds.

2 Preliminary Notes

A smooth manifold together with an affine connection ∇ is called an equi-affine structure whenever, there exist a parallel volume element ω on M. That is for every $p \in M$ there exists an open neighborhood U such that $\nabla\omega = 0$ on U. Usually (∇, ω) is called compatible, and (M, ∇, ω) is locally equi-affine manifold.

Example 2.1 *The Euclidian space $\mathbb{R}^n$ together with usual connection D and usual volume element is an equi-affine manifold.*

Example 2.2 *A Riemannian manifold M together with Levi-Civita connection and corresponding volume element is an equi-affine manifold.*[5, pp.362]

Definition 2.3 *The diffeomorphism $f : (M, \nabla) \to (\bar{M}, \bar{\nabla})$ is an affine transformation whenever $f_*(\nabla_X Y) = \bar{\nabla}_{f_*X} f_*Y$, hence*

$$f : (M, \nabla, \omega) \to (\bar{M}, \bar{\nabla}, \bar{\omega})$$

is equi-affine if it is affine and keeps the volume form ω invariant i.e $f^*\bar{\omega} = \omega$.

If $\mathcal{B} = \{e_1, e_2, \cdots, e_n\}$ is a base for T_pM, then $\mathcal{B}$ is equi-affine or (unimodular) whenever $\omega_p(e_1, e_2, \cdots, e_n)$=1. Let V be a real $n-$dimensional vector space and ω be a volume form on V, then the vectors $x_1, \cdots, x_n$ in V are linearly independent iff $\omega(x_1, \cdots, x_n) = \ 0$.

Proposition 2.4 *Let (M, ∇) be affine structure of dimension n.
If $V_1(t), \ldots, V_n(t)$ are vector fields defined along a curve α and parallel relative to ∇ and ω a parallel volume form, then $\omega(V_1(t), \ldots, V_n(t))$ is constant.*[8]

Proposition 2.5 *Let M be a manifold, and ∇ be an torsion-free affine connection on M then the following conditions are equivalent:*
1) ∇ is locally equi-affine ;
2) Ric is symmetric.[8, pp.14]

3 Equi-affine vector field

Let $U \subseteq M$ be a connected neighborhood on equi-affine structure (M, ∇, ω).

Definition 3.1 *Let (M, ∇, ω) be an equi-affine structure of dimension two and X, Y be two vector fields on U, the vector field Y is an equi-affine vector field in X direction, whenever $\omega(Y, \nabla_X Y) = 1$. A vector field X is equi-affine if it is equi-affine in its direction.*
We extend the above definition along any curve α on M . That is Y is equi-affine along α if $\omega(Y, \frac{\nabla Y}{dt}) = 1$ .We extend for manifolds of dimension $n > 2$ .Let M be a manifold of dimension $n > 2$, and (M, ∇, ω) an equi-affine structure. Let X, Y be vector fields on $U \subseteq M$, then Y is equi-affine in X direction if

$$\omega(Y, \nabla_X Y, \nabla_X^2 Y, \dots, \nabla_X^{n-1} Y) = 1. \tag{1}$$

Also the vector field Y will be called an absolutely equi-affine vector field in X direction on U. whenever absolute value of left side of (1) is equal to 1.

Definition 3.2 *The vector field Y is called nondegenerate(positive nondegenerate) in X direction if the left side of (1) is nonzero(positive). The curve α is nondegenerate(positive nondegenerate) if $X = \alpha'$ is nondegenerate(positive nondegenerate) vector field,along α.*

Example 3.3 *Let $M = \mathbb{R}^2$ be with (x, y) coordinate, and $X = \frac{\partial}{\partial x}, Y_1 = x\frac{\partial}{\partial x} + (x^2 - y^2)\frac{\partial}{\partial y}, Y_2 = (x/y)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ be local vector fields on $U = \{(x, y) | x, y \in \mathbb{R}, y > 0\}$, then Y_1, Y_2 are positive nondegenerate and absolutely equi-affine vector fields on U,in X direction respectively.*

Theorem 3.4 *In any equi-affine structure, there exist absolutely equi-affine vector fields in any direction.*

Proof : Let $U \subseteq M$ be a connected neighborhood of an equi-affine structure (M, ∇, ω) of dimension n, and Y_0 be a nondegenerate vector field in X direction on U. define a positive map $f : U \to \mathbb{R}$ such that $Y = fY_0$ is an absolutely equi-affine vector field in X direction. We have the relation

$$\nabla_X^k (fY_0) = \sum_{r=0}^{k} \binom{k}{r} X^{k-r}(f) \nabla_X^r Y_0, \tag{2}$$

where $\nabla^0_X Y_0 = Y_0, X^0(f) = f$. Since Y is absolutely equi-affine in X direction, hence satisfies in definition (3.1), and we have

$$f = \ |\omega(Y_0, \nabla_X Y_0, \dots, \nabla_X^{n-1} Y_0)|^{-1/n}.\Box$$

Theorem 3.5 *There exists absolutely equi-affine vector field on any manifold M with equi-affine structure .*

Proof : Let (M, ∇, ω) be an equi-affine structure of dimension n, and X_0 a nondegenerate vector field on connected neighborhood of $U \subseteq M$. It suffice to find a positive function $f : U \to \mathbb{R}$ such that $X = fX_0$, hence an absolutely equi-affine vector field on U. We should have

$$|\omega(fX_0, \nabla_{fX_0}(fX_0), \dots, \nabla^{n-1}_{fX_0} fX_0)| = 1,$$
$$f^{n(n+1)/2}|\omega(X_0, \nabla_{X_0} X_0, \dots, \nabla^{n-1}_{X_0} X_0)| = 1,$$
$$f = |\omega(X_0, \nabla_{X_0} X_0, \dots, \nabla^{n-1}_{X_0} X_0)|^{-2/n(n+1)}. \Box$$

Definition 3.6 *The parameter of curve α on M is called affine arclength parameter if α' is an absolutely equi-affine vector field along α.*

Definition 3.7 *Let $\alpha : I \to M$ and $\beta : J \to M$ be differentiable curves, the curve β is said reparametrization of α provided there exists a diffeomorphism $h : J \to I$ such that $\beta = \alpha o h$ and $h'(t) > 0$ for all $t \in J$.*

Theorem 3.8 *In any equi-affine structure of dimension n , every nondegenerate curve can be reparametrization by arclength .*

Proof : Let $\alpha : I \to M$ be a nondegenerate curve on M with equi-affine structure. We suppose

$$s = \int_{t_0}^{t} |\omega(\alpha'(t), \alpha''(t), \cdots, \alpha^{(n)}(t))|^{1/p}\, dt \ \ , p = n(n+1)/2,$$

therefore

$$\frac{ds}{dt} = |\omega(\alpha'(t), \alpha''(t), \cdots, \alpha^{(n)}(t))|,$$

since $\frac{ds}{dt} > 0$ hence $t = h(s)$. We define $\beta = \alpha o h$ then the curve β is reparametrization by affine arclength of α. $\Box$

Remark 3.9 *The image of a curve with affine arclength parameter under an equi-affine transformation is a curve with affine arclength parameter.*

Definition 3.10 *Let $\alpha : [a, b] \to M$ be a nondegenerate curve on M with equi-affine structure,*

$$d(a, b) := \int_a^b |\omega(\alpha'(t), \alpha''(t), \cdots, \alpha^{(n)}(t))|^{2/n(n+1)}\, dt,$$

is called the affine arc length of the curve α.

Remark 3.11 *In any equi-affine structure, affine arclength is invariant under any equi-affine transformation.*

4 Affine curvatures of curves on 2-manifolds

Let M be a two dimensional manifold and (M, ∇, ω) an equi-affine structure.

Definition 4.1 *Let Y be an equi-affine vector field on $U \subseteq M$ in X direction,hence we have $\omega(Y, \nabla_X^2 Y) = 0$ or $\nabla_X^2 Y = -\mathfrak{K}_X(Y)Y$, where $\mathfrak{K}_X(Y)$ is a real number related to X and Y. If Y is an equi-affine vector field along α, we will denote $\mathfrak{K}_X(Y)$ by $\mathfrak{K}(Y)$ that is $\nabla^2 Y/dt^2 = -\mathfrak{K}(Y)Y$. If α is a curve with affine arclength parameter on M, then $\nabla^2\alpha'/dt^2 = -\kappa_a\alpha'$.*

Definition 4.2 *Let α be a curve with affine arclength parameter on M, $\kappa_a := \mathfrak{K}(\alpha')$ is called the affine curvature of the curve α.*

Proposition 4.3 *In any equi-affine structure (M, ∇, ω) of dimension 2, $\mathfrak{K}_X(Y)$ is calculated as*

$$\mathfrak{K}_X(Y) = \omega(\nabla_X Y, \nabla_X^2 Y). \tag{3}$$

Proof : It is straight forward from the definition (4.1).

Remark 4.4 *The necessary and sufficient condition for $\nabla_X Y$ to be an equi-affine vector field in direction X is $\mathfrak{K}_X(Y) = 1$.*

Remark 4.5 *In any equi-affine structure (M, ∇, ω) the affine curvature of a curve with affine arclength parameter is calculated by*

$$\kappa_a = \omega(\frac{\nabla\alpha'}{dt}, \frac{\nabla^2\alpha'}{dt^2}).$$

Proposition 4.6 *Let $(M, \nabla, \omega), (\bar{M}, \bar{\nabla}, \bar{\omega})$ be two equi-affine structure and $f : M \to \bar{M}$ be an equi-affine transformation , then $\bar{\mathfrak{K}}_{f_*X}(f_*Y) = \mathfrak{K}_X(Y)$.*

Proof : we get from (3) that

$$\begin{aligned}
\bar{\mathfrak{K}}_{f_*X}(f_*Y) &= \bar{\omega}(\bar{\nabla}_{f_*X} f_*Y, \bar{\nabla}^2_{f_*X} f_*Y) \\
&= \bar{\omega}(f_*\nabla_X Y, f_*\nabla_X^2 Y) \\
&= \omega(\nabla_X Y, \nabla_X^2 Y) \\
&= \mathfrak{K}_X(Y). \ \square
\end{aligned}$$

Remark 4.7 *In any equi-affine structure, $\mathfrak{K}_X Y$ is invariant under any equi-affine transformation.*

Remark 4.8 *Affine curvature of any curve is invariant under any equi-affine transformation.*

Let $\{E_1, E_2\}$ be an equi-affine parallel base along α, that is $\omega(E_1(t), E_2(t)) = 1$ for all $t \in I$, and $Y(t)$ is a vector field along α, then

$$\begin{aligned} &Y = y_1 E_1 + y_2 E_2, \\ &\nabla Y/dt = y_1' E_1 + y_2' E_2, \\ &\nabla^2 Y/dt^2 = y_1'' E_1 + y_2'' E_2. \end{aligned}$$

In the base $\{E_1, E_2\}$ we can write the differential equation $\nabla^2 Y/dt^2 = -\mathfrak{K}(Y)Y$ as:

$$\begin{cases} y_1'' + \mathfrak{K}(Y) y_1 = 0, \\ y_2'' + \mathfrak{K}(Y) y_2 = 0. \end{cases} \tag{4}$$

If $V(t), W(t)$ are two equi-affine parallel vector fields along the differentiable curve α , then the vector fields

$$\begin{aligned} &Y_1(t) = tV(t) + W(t), \\ &Y_2(t) = \cos t\, V(t) + \sin t\, W(t), \\ &Y_3(t) = \cosh t\, V(t) + \sinh t\, W(t), \\ &Y_4(t) = 1/t\ V(t) + 1/3\ t^2 W(t). \end{aligned}$$

are equi-affine along α .
By the equation (3) we get $\mathfrak{K}(Y_1) = 0, \mathfrak{K}(Y_2) = 1, \mathfrak{K}(Y_3) = -1, \mathfrak{K}(Y_4) = -14/3\, t^{-2}$.
In the proposition (4.3) we see that for a vector field Y along $\alpha, \mathfrak{K}(Y)$ is deduced from (3). The question arises that for a given function $\mathfrak{K}(t)$ is there any equi-affine vector field along α such that $\mathfrak{K}(Y) = \mathfrak{K}(t)$?

Theorem 4.9 *Let (M, ∇, ω) be a two dimensional equi-affine structure and $\alpha : I \to M, k : I \to \mathbb{R}$ differential maps and v_0, w_0 two vectors at $p = \alpha(t_0)$ with $\omega(v_0, w_0) = 1$ then there exist a unique equi-affine vector field Y along α such that $\mathfrak{K}(Y) = k(t)$, $Y(t_0) = v_0$, $\frac{\nabla Y}{dt}(t_0) = w_0$.*

Proof : Let E_1, E_2 be parallel translation of equi-affine base along α. The components (y_1, y_2) of $Y(t)$ will satisfy in (4). By the equi-affine condition for $Y(t)$ we get $\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \in SL(2, R)$, and by the existence and uniqueness conditions there exist solutions for y_1 and y_2 with initial conditions $y_1(t_0) = 1, y_1'(t_0) = 0$, $y_2(t_0) = 0, y_2'(t_0) = 1$. $\square$

Theorem 4.10 *Let (M, ∇, ω) be a two dimensional equi-affine structure and $\alpha : I \to M$ be differential curve on M, let v_0, w_0 be vectors on $\alpha(0) = p$*

with $\omega(v_0, w_0) = 1$*, then any equi-affine vector field* Y *with* $\mathfrak{K}(Y) = k$ *(constant) will be expressed as*

$$Y_1(t) = tV(t) + W(t), \qquad k = 0, \tag{5}$$

$$Y_2(t) = \cos(\sqrt{k}\,t)V(t) + \frac{\sin(\sqrt{k}\,t)}{\sqrt{k}}W(t), \qquad k > 0, \tag{6}$$

$$Y_3(t) = \cosh(\sqrt{-k}\,t)V(t) + \frac{\sinh(\sqrt{-k}\,t)}{\sqrt{-k}}W(t), \qquad k < 0. \tag{7}$$

where $V(t), W(t)$ *are parallel transformations of* v_0, w_0 *along* α*.*

Proof : Let $V(t), W(t)$ be parallel translations of v_0, w_0 along α. Then $\{V(t), W(t)\}$ will be an equi-affine base along α. By the proof of theorem (4.9) and the equations

$$\begin{cases} y_1'' + ky_1 = 0, \\ y_1(0) = 1, y_1'(0) = 0, \end{cases} \qquad \begin{cases} y_2'' + ky_2 = 0, \\ y_2(0) = 0, y_2'(0) = 1. \end{cases}$$

the characteristic equations for both is $r^2 + k = 0$, hence

Case: 1 *if* $k = 0$*, then* $y_2(t) = 1$, $y_1(t) = t$*;*

Case: 2 *if* $k > 0$*, then* $y_2(t) = \frac{\sin(\sqrt{k}\,t)}{\sqrt{k}}$, $y_1(t) = \cos(\sqrt{k}\,t)$*;*

Case: 3 *if* $k < 0$*, then* $y_2(t) = \frac{\sinh(\sqrt{-k}\,t)}{\sqrt{-k}}$, $y_1(t) = \cosh(\sqrt{-k}\,t)$. □

5 Affine curvature of curve on n-manifold

In this section, let M be an n-dimensional manifold and (M, ∇, ω) an equi-affine structure . As in definition (3.1) a vector field Y is equi-affine along X whenever the equation (1) is satisfied . We can deduce that,

$$\omega(Y, \nabla_X Y, \nabla_X^2 Y, \ldots, \nabla_X^{n-2} Y, \nabla_X^n Y) = 0,$$

hence we get

$$\nabla_X^n Y = -\mathfrak{K}_X^1(Y)Y - \mathfrak{K}_X^2(Y)\nabla_X Y - \ldots - \mathfrak{K}_X^{n-1}(Y)\nabla_X^{n-2} Y, \tag{8}$$

therefore we can define.

Definition 5.1 *Let* α *be a curve with affine arclength parameter, then* $\kappa_a^i := \mathfrak{K}^i(\alpha')$ $(i = 1, \ldots, n-1)$ *is called* i*-th affine curvature of the curve* α*.*

Theorem 5.2 *In any equi-affine structure* (M, ∇, ω) *of dimension* n , $\mathfrak{K}_X^i(Y)$ *can be calculated as*

$$\mathfrak{K}_X^i(Y) = -\omega(\nabla_X^{\sigma(0)} Y, \ldots, \nabla_X^{\sigma(n-1)} Y), \tag{9}$$

where σ *is a permutation on* $\{0, 1, \cdots, n\}$ *such that* $\sigma(i-1) = n$; $i = 1, \cdots, n-1$, $\sigma(k) = k$; $k \neq i-1, n, k \in \{0, 1, \cdots, n\}$.

Proof : Let σ be a permutation on $\{0, 1, \cdots, n\}$ such that $\sigma(i-1) = n$; $i = 1, \cdots, n-1$, $\sigma(k) = k$; $k \neq i-1, n, k \in \{0, 1, \cdots, n\}$. Let $\nabla_X^n Y$ and $\nabla_X^{j-1} Y$ be i-th and j-th component, respectively where $j \neq i$, $j = 1, \cdots, n-1, k = j-1$.By the equation (1),(8), we can deduce the equation (9). □

Theorem 5.3 *Let* $(M, \nabla, \omega), (\bar{M}, \bar{\nabla}, \bar{\omega})$ *be two equi-affine structure of dimension* n , *and* $f : M \to \bar{M}$ *be an equi-affine transformation , then*

$$\bar{\mathfrak{K}}_{f_*^i X}(f_* Y) = \mathfrak{K}_X^i(Y) \; ; \; i = 1, \cdots, n-1.$$

Proof : By the equation (9) and affinity of f, It is obvious. □

Remark 5.4 *The* i*-th equi-affine curvature are invariant under the equi-affine translations.*

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