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New Result on the Existence of Positive Periodic Solutions to Neutral Population Model with Multiple Delays[1]

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Abstract

In this paper, the authors employ an abstract continuous theorem of k-set contractive operator and some analysis techniques to study a neutral population model with multiple delays as follows

$$\frac{dN}{dt} = N(t)[a(t) - \beta(t)N(t) - \sum_{j=1}^{n} b_j(t)N(t-\sigma_i(t)) - \sum_{i=1}^{m} c_i(t)N'(t-\tau_i(t))].$$

A new result on the existence of positive periodic solutions is obtained.

Mathematics Subject Classification: 34C25, 34K40

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§1. Introduction

Let $\omega > 0$ be a constant, $C_\omega = \{x | x \in C(R, R), x(t + \omega) \equiv x(t)\}$ with the norm defined by $|x|_0 = \max_{t\in[0,\omega]} |x(t)|$, and $C^1_\omega = \{x|x \in C^1(R, R), x(t+\omega) \equiv x(t)\}$ with the norm defined by $||x|| = \max\{|x|_0, |x'|_0\}$. Then C_ω, C^1_ω are all Banach spaces, and also we denote $\bar{h} := \frac{1}{\omega}\int_0^\omega h(s)ds$.

In the past few years, the problems of positive periodic solutions for some Lotka Volterra population models with delay were studied by [1-4]. Also, there

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were some work, see [5-7] and the references therein, to study the problems of boundedness and stability of solutions to some neutral population models with delay. In 1993, Kuang Yang in paper [8] proposed an open problem(open problem 9.2) to obtain sufficient conditions for the existence of positive periodic solutions of the following equation

$$\frac{dN}{dt} = N(t)[a(t) - \beta(t)N(t) - b(t)N(t-\tau(t)) - c(t)N'(t-\tau(t))], \qquad (1.1)$$

where $a(t), \beta(t), b(t), c(t)$ and $\tau(t)$ are all in C_ω with $a(t) \geq 0, \tau(t) \geq 0, \forall t \in [0,\omega]$. In recent years, some researchers have paid much attention to such an open problem. For example, Fang Hui and Li Jibin in paper[9] studied Eq.(1.1) and gave an answer to open problem 9.2 of paper [8], and also in [10], Lu and Ge consider the existence of positive periodic solutions for the following neutral population model with multiple delays

$$\frac{dN}{dt} = N(t)[a(t)-\beta(t)N(t)-\sum_{j=1}^{n} b_j(t)N(t-\sigma_j(t))-\sum_{i=1}^{m} c_i(t)N'(t-\tau_i(t))], \quad (1.2)$$

where $a(t), \beta(t), b_j(t), c_i(t), \sigma_j(t), \tau_i(t)$ are all in C_ω with $\sigma_j(t) \geq 0$ and $\tau_i(t) \geq 0, \forall t \in [0,\omega]$. Furthermore, $\sigma_j \in C^1(R,R)$ with $\sigma_j'(t) < 1, \forall t \in [0,\omega]$, $c_i \in C^1(R,R)$ and $\tau_i(t) \in C^2(R,R)$ with $\tau_i'(t) < 1, \forall t \in [0,\omega], \forall i \in \{1,2,\cdots,m\}, \forall j \in \{1,2,\cdots,n\}$. But the crucial condition such as

$[C]$ There is a constant $r_0 > 0$ such that

$$\max\{\sum_{i=1}^{m} |c_i|_0, \sqrt{2}\sum_{j=1}^{n} |b_j|_0 l_j + \sum_{i=1}^{m} |c_{0,i}|_0\} e^{r_0} < 1 \qquad (1.3)$$

and

$$r_0 > \left|\ln \frac{\bar{a}}{\bar{\beta} + \sum_{j=1}^{n} \bar{b}_j}\right| + \omega^{1/2} \left(\frac{r_0|a'|_1 + [|\beta'|_1 + \sum_{j=1}^{n} |b'|_1]e^{r_0}}{1 - [\sqrt{2}\sum_{j=1}^{n} |b_j|_0 l_j + \sum_{i=1}^{m} |c_{0,i}|_0]e^{r_0}}\right)^{1/2},$$

where l_j is a constant with $l_j = \min_{m_j \in Z} |\sigma_j - m_j\omega|_0$, Z is the set of integer, $c_{0,i}(t) = \frac{c_i(t)}{1-\tau_i'(t)}$, $i = 1,2,\cdots,m, j = 1,2,\cdots,n$, was required.

In this paper, we continue to study the existence of positive periodic solutions to Eq. (1.2). By using an abstract continuation theorem for k-set contraction[11,12] and some analysis techniques, a new result on the existence of positive periodic solutions is obtained. The interest is that the conditions guaranteed the existence of positive periodic solutions to Eq.(1.2) are different from the corresponding ones of [10]. For example, we require

$$\max\{\sum_{i=1}^{m} |c_i|_0, \sum_{i=1}^{m} |c_{0,i}|_0\} e^{r_0} < 1,$$

which is weaker than condition (1.3).

In order to study the problem of existence of positive periodic solution to Eq.(1.2), we take the substitution $N(t)=e^{x(t)}$, and then (1.2) can be rewritten in the following form

$$x'(t)=a(t)-\beta(t)e^{x(t)}-\sum_{j=1}^{n}b_j(t)e^{x(t-\sigma_j(t))}-\sum_{i=1}^{m}c_i(t)x'(t-\tau_i(t))e^{x(t-\tau_i(t))}. \quad (1.4)$$

Remark 1.1 From the substitution $N(t)=e^{x(t)}$, it is easy to see that Eq.(1.2) has a positive periodic solution $N(t)=e^{y(t)}$ if and only if Eq.(1.4) has a periodic solution $x(t)=y(t)$.

§2. Main Lemmas

For the sack of applying an abstract continuous theorem of k-set contractive operator to study Eq.(1.4), we should make some preparations in the fist for the convenience of the reader.

Let E be a Banach space. For a bounded subset$A\subset E$,

$$\begin{aligned}\alpha_E(A) &= \inf\{\delta>0 | \text{There is a finite number of subsets} \\ &\quad A_i\subset A \text{ such that } A=\cup_{i=1}A_i \text{ and } \mathrm{d}iam(A_i)\le\delta\}\end{aligned}$$

denote the (Kuratoskii) measure of non-compactness, where $\mathrm{d}iam(A_i)$ denotes the diameter of set A_i. Let X, Y be two Banach spaces and Ω be a bounded open subset of X. A continuous and bounded map $\mathcal{N}:\bar{\Omega}\to Y$ is called k-set contractive if for any bounded set $A\subset\Omega$ we have

$$\alpha_Y(\mathcal{N}(A))\le k\alpha_X(A),$$

where k is a constant. Also, for a Fredholm operator $Ł: X\to Y$ with index zero, according to papers [11,12] we may define that

$$l(Ł)=\sup\{r\ge 0 | r\alpha_X(A)\le\alpha_Y(Ł(A)), \text{for all bounded subset } A\subset X\}.$$

Lemma 2.1[13]. Let $Ł: X\to Y$ be a Fredholm operator with zero index, and $a\in Y$ be a fixed point. Suppose that $\mathcal{N}:\Omega\to Y$ is a $k-$set contractive with $k<l(Ł)$, where $\Omega\subset X$ is bounded, open, and symmetric about $0\in\Omega$. Further, we also assume that

(1) $Łx\ne\lambda\mathcal{N}x+\lambda a,$ for $x\in\partial\Omega, \lambda\in(0,1)$, and

(2) $[Q\mathcal{N}(x)+Qa,x]\cdot[Q\mathcal{N}(-x)+Qa,x]<0,$ for $x\in kerŁ\cap\partial\Omega$

where $[.,.]$ is a bilinear form on $Y\times X$ and Q is the projection of Y onto $Coker(Ł)$, where $Coker(Ł)$ is the cokernel of the operator Ł. Then there is a $x\in\overline{\Omega}$ such that

$$Łx-\mathcal{N}x=a.$$

In order to use Lemma 2.1 to study Eq. (1.4), we set $Y = C_\omega$, $X = C^1_\omega$,

$$Ł : X \to Y,\ Łx = \frac{dx}{dt} \tag{2.1}$$

and $\mathcal{N} : X \to Y$ defined by

$$\mathcal{N}x = -\beta(t)e^{x(t)} - \sum_{j=1}^{n} b(t)e^{x(t-\sigma_j(t))} - \sum_{i=1}^{m} c_i(t)x'(t-\tau_i(t))e^{x(t-\tau_i(t))}. \tag{2.2}$$

Remark 2.1: Clearly, Eq.(1.4) can be converted to$Łx - \mathcal{N}x = a$ for $a := a(t)$.

Lemma 2.2[Lemma 3.2, 9, 10] The differential operator Ł is a Fredholm operator with index zero, and satisfies $l(Ł) \geq 1$.

Lemma 2.3. Let r_0, r_1 be two positive constants, and $\Omega = \{x | x \in C^1_\omega, |x|_0 < r_0, |x'|_0 < r_1\}$. If $k = (\sum\limits_{i=1}^{m} |c_i|_0)e^{r_0}$, then $N : \Omega \to C_\omega$ is a k-contractive map.

As Lemma 2.3 can be proved in the same way as in the proofs of Lemma 3 of [14], we omit it here.

Lemma 2.4 [Lemma 2.4, 13] Suppose $\tau \in C^1_\omega$ and $\tau'(t) < 1, \forall t \in [0,\omega]$. Then the function $t - \tau(t)$ has a unique inverse $\mu(t)$ satisfying $\mu \in C(R,R)$ with $\mu(u+\omega) = \mu(u) + \omega, \forall u \in R$.

Remark 2.2: By using Lemma 2.4, we see that if $g \in C_\omega$, $\tau \in C^1_\omega$ and $\tau'(t) < 1, \forall t \in [0,\omega]$, then $g(\mu(t+\omega)) = g(\mu(t)+\omega) = g(\mu(t)), \forall t \in R$, where $\mu(t)$ is the inverse function of $t-\tau(t)$, which together with $\mu \in C(R,R)$ implies that $g(\mu(t)) \in C_\omega$.

§3.Main Result

As $\tau_i'(t) < 1, \sigma_j'(t) < 1, \forall t \in [0,\omega]$, we see that either $t - \tau_i(t)$ or $t - \sigma_j(t)$ has a unique inverse. Throughout this paper, we set $\gamma_i(t), \mu_j(t)$ to represent the inverse of function $t - \tau_i(t)$ and $t - \sigma_j(t) (i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$, respectively. Furthermore

$$\Gamma(t) := \beta(t) + \sum_{j=1}^{n} \frac{b_j(\mu_j(t))}{1 - \sigma_j'(\mu_j(t))} - \sum_{i=1}^{m} \frac{c_{0,i}'(\gamma_i(t))}{1 - \tau_i'(\gamma_i(t))}, \tag{3.1}$$

where $c_{0,i}(t) = \frac{c_i(t)}{1-\tau_i'(t)}$, $(i = 1, 1, \cdots, m)$, and

$$\Gamma_1(t) := |\beta(t)| + \left| \sum_{j=1}^{n} \frac{b_j(\mu_j(t))}{1 - \sigma_j'(\mu_j(t))} \right|. \tag{3.2}$$

Remark 3.1: By Lemma 2.4, we see $\mu_j(\omega) = \mu_j(0) + \omega, \forall j \in \{1, 2, \cdots, n\}$. So

$$\int_0^\omega \frac{b_j(\mu_j(s))}{1 - \sigma_j'(\mu_j(s))} ds = \int_{\mu_j(0)}^{\mu_j(\omega)} \frac{b_j(t)(1 - \sigma_j'(t))}{1 - \sigma_j'(t)} dt = \int_{\mu_j(0)}^{\mu_j(0)+\omega} b_j(t) dt = \bar{b}_j,$$

$$\forall j \in \{1, 2, \cdots, n\}.$$

Similarly,

$$\int_0^\omega \frac{c'_{0,i}(\gamma_i(s))}{1-\tau'_i(\gamma_i(s))}ds = \int_{\gamma_i(0)}^{\gamma_i(\omega)} \frac{c'_{0,i}(t)(1-\tau'_i(t))}{1-\tau'_i(t)}dt = \int_0^\omega c'_{0,i}(t)dt = 0, \forall i \in \{1, 2, \cdots, m\}.$$

Thus,

$$\begin{aligned}\overline{\Gamma} = \frac{1}{\omega}\int_0^\omega \Gamma(s)ds &= \frac{1}{\omega}\left[\beta(t) + \sum_{j=1}^{n}\int_0^\omega \frac{b_j(\mu_j(s))}{1-\sigma'_j(\mu_j(s))}ds - \sum_{i=1}^{m}\int_0^\omega \frac{c'_{0,i}(\gamma_i(s))}{1-\tau'_i(\gamma_i(s))}ds\right] \\ &= \bar{\beta} + \sum_{j=1}^{n}\bar{b}_j. \qquad (3.3)\end{aligned}$$

Theorem 3.1 Suppose that the following conditions hold.
$[H_1]$ $\bar{a} > 0, \Gamma(t) > 0, \forall t \in [0, \omega]$.
$[H_2]$ There is a constant $r_0 > 0$ such that

$$\max\{\sum_{i=1}^{m}|c_i|_0, \sum_{i=1}^{m}|c_{0,i}|_0\}e^{r_0} < 1$$

and

$$r_0 > |\ln\frac{\bar{a}}{\bar{\beta} + \sum_{j=1}^{n}\bar{b}_j}| + \frac{\omega\overline{|a|}[1 + |\frac{\Gamma_1}{\Gamma}|_0]}{1 - \sum_{i=1}^{m}|c_{0,i}|_0 e^{r_0}}.$$

Then Eq.(1.2) has at least one positive $\omega-$ periodic solution, where $\Gamma(t)$ and $\Gamma_1(t)$ is defined by (3.1) and (3.2), respectively.

Proof: From Remark 1.1 and Remak 2.1, it suffice to show that the equation $Lx = \mathcal{N}x + a$ has a solution $x \in C_\omega$, where $a := a(t)$, $\mathcal{N}, Ł$ is defined by (2.1) and (2.2), respectively. In order to do it, let $u(t)$ be an arbitrary $\omega-$periodic solution of the operator equation as follows

$$Łu = \lambda\mathcal{N}u + \lambda a, \lambda \in (0, 1).$$

Then

$$u'(t) = \lambda[a(t) - \beta(t)e^{u(t)} - \sum_{j=1}^{n}b_j(t)e^{u(t-\sigma_j(t))} - \sum_{i=1}^{m}c_i(t)u'(t-\tau_i(t))e^{u(t-\tau_i(t))}]. \qquad (3.4)$$

Integrating both sides of (3.4) over $[0, \omega]$, we have

$$\bar{a}\omega = \int_0^\omega[\beta(t)e^{u(t)} + \sum_{j=1}^{n}b_j(t)e^{u(t-\sigma_j(t))} - \sum_{i=1}^{m}c'_{0,i}(t)e^{u(t-\tau_i(t))}]dt. \qquad (3.5)$$

Let $t-\sigma_j(t)=s$, i.e., $t=\mu_j(s)$, then

$$\int_0^\omega b_j(t)e^{u(t-\sigma_j(t))}dt=\int_{-\sigma_j(0)}^{\omega-\sigma_j(\omega)}\frac{b_j(\mu_j(s))}{1-\sigma_j'(\mu_j(s))}e^{u(s)}ds.$$

In view of Remark 2.1, we see

$$\int_0^\omega b_j(t)e^{u(t-\sigma_j(t))}dt=\int_0^\omega\frac{b_j(\mu_j(s))}{1-\sigma'(\mu_j(s))}e^{u(s)}ds,(j=1,2,\cdots,n).$$

Similarly,

$$\int_0^\omega c_{0,i}'(t)e^{u(t-\tau_i(t))}dt=\int_0^\omega\frac{c_{0,i}'(\gamma_i(s))}{1-\tau'(\gamma_i(s))}e^{u(s)}ds,(i=1,2,\cdots,m).$$

So from (3.5), we get

$$\int_0^\omega\Gamma(s)e^{u(s)}ds=\bar{a}\omega. \tag{3.6}$$

Considering assumption $[H_1]$, we know $\Gamma(t)\geq 0,\forall t\in[0,\omega]$, and then it follows from the integro mean value theorem that there is a $\xi\in[0,\omega]$ such that

$$e^{u(\xi)}\int_0^\omega\Gamma(s)ds=\bar{a}\omega$$

i.e.,

$$e^{u(\xi)}=\frac{\bar{a}}{\bar{\bar{\Gamma}}},$$

which together with (3.3) yields

$$u(\xi)=\ln\frac{\bar{a}}{\bar{\bar{\Gamma}}}=\frac{\bar{a}}{\bar{\beta}+\sum_{j=1}^n\bar{b}_j},$$

and then

$$|u|_0\leq|\ln\frac{\bar{a}}{\bar{\beta}+\sum_{j=1}^n\bar{b}_j}|+\int_0^\omega|u'(s)|ds. \tag{3.7}$$

Let $\Omega=\{x:x\in C_\omega^1,|x|_0<r_0,|x'|_0<r_1\}$, where r_0 is defined by $[H_2]$ and r_1 is a constant satisfying

$$r_1>\frac{|a|_0+|\beta|_0+\sum_{j=1}^n|b_j|_0}{1-\sum_{i=1}^m|c_i|_0e^{r_0}}. \tag{3.8}$$

In what follows, we will prove that

$$\L x\neq\lambda\mathcal{N}x+\lambda a,\forall\lambda\in(0,1)\text{ and }\forall x\in\partial\Omega.$$

Suppose the contrary. Then there must be a $\lambda\in(0,1)$ and an $x\in\partial\Omega$ such that

$$\L x=\lambda\mathcal{N}x+\lambda a,$$

i.e.,

$$x'(t) = \lambda[a(t) - \beta(t)e^{x(t)} - \sum_{j=1}^{n} b_j(t)e^{x(t-\sigma_j(t))} - \sum_{i=1}^{m} c_i(t)x'(t-\tau_i(t))e^{x(t-\tau_i(t))}]. \tag{3.9}$$

From (3.7), we see

$$|x|_0 \le |\ln \frac{\bar{a}}{\bar{\beta} + \sum_{j=1}^{n} \bar{b}_j}| + \int_0^\omega |x'(s)|ds. \tag{3.10}$$

In view of $x \in \partial\Omega$, we see either $|x|_0 = r_0$ and $|x'|_0 \le r_1$; or $|x'|_0 = r_1$ and $|x|_0 \le r_0$.

Case 1. If $|x|_0 = r_0$, then from (3.9) we get

$$\begin{aligned}
& \int_0^\omega |x'(t)|dt \\
\le\ & \int_0^\omega |a(t)|dt + \int_0^\omega |\beta(t)|e^{x(t)}dt + \sum_{j=1}^{n} |b_j(t)|e^{x(t-\sigma_j(t))}dt + \\
& e^{|x|_0} \sum_{i=1}^{m} \int_0^\omega |c_i(t)||x'(t-\tau_i(t))|dt \\
=\ & \overline{|a|}\omega + \int_0^\omega |\beta(t)|e^{x(t)}dt + \sum_{j=1}^{n} \int_0^\omega \frac{|b_j(\mu_j(s))|}{1-\sigma_j'(\mu_j(s))} e^{x(s)}ds + \\
& e^{|x|_0} \sum_{i=1}^{m} \int_0^\omega |c_{0,i}(s)x'(s)|ds \\
\le\ & \overline{|a|}\omega + \int_0^\omega \Gamma_1(t)e^{x(t)}dt + e^{|x|_0} \sum_{i=1}^{m} |c_{0,i}|_0 \int_0^\omega |x'(s)|ds \\
\le\ & \overline{|a|}\omega + \int_0^\omega \frac{\Gamma_1(t)}{\Gamma(t)} \Gamma(t)e^{x(t)}dt + e^{|x|_0} \sum_{i=1}^{m} |c_{0,i}|_0 \int_0^\omega |x'(s)|ds \\
\le\ & \overline{|a|}\omega + |\frac{\Gamma_1}{\Gamma}|_0 \int_0^\omega \Gamma(t)e^{x(t)}dt + e^{|x|_0} \sum_{i=1}^{m} |c_{0,i}|_0 \int_0^\omega |x'(s)|ds.
\end{aligned}$$

By (3.6) and $|x|_0 = r_0$, we see from the above inequality that

$$\int_0^\omega |x'(t)|dt \le \overline{|a|}\omega[1 + |\frac{\Gamma_1}{\Gamma}|_0] + e^{|r|_0} \sum_{i=1}^{m} |c_{0,i}|_0 \int_0^\omega |x'(s)|ds. \tag{3.11}$$

As $e^{|r|_0} \sum_{i=1}^{m} |c_{0,i}|_0 < 1$, it follows from (3.11) that

$$\int_0^\omega |x'(t)|dt \le \frac{\overline{|a|}\omega[1 + |\frac{\Gamma_1}{\Gamma}|_0]}{1 - e^{|r|_0} \sum_{i=1}^{m} |c_{0,i}|_0}.$$

So by (3.10), we have

$$r_0 = |x|_0 \le |\ln \frac{\bar{a}}{\bar{\beta} + \sum_{j=1}^{n} \bar{b}_j}| + \frac{\overline{|a|}\omega[1 + |\frac{\Gamma_1}{\Gamma}|_0]}{1 - e^{|r|_0} \sum_{i=1}^{m} |c_{0,i}|_0},$$

which contradicts assumption $[H_2]: r_0 > |\ln\frac{\bar a}{\bar\beta+\sum_{j=1}^n \bar b_j}| + \frac{\overline{|a|}\omega[1+|\frac{\Gamma_1}{\Gamma}|_0]}{1-e^{|r|_0}\sum_{i=1}^m |c_{0,i}|_0}$.

Case 2. If $|x'|_0 = r_1$ and $|x|_0 \le r_0$, then by (3.7) we see

$$\begin{aligned} |x'|_0 &\le |a|_0 + |\beta|_0 e^{|x|_0} + \sum_{j=1}^n |b_j|_0 e^{|x|_0} + \sum_{i=1}^m |c_i|_0 e^{|x|_0}|x'|_0 \\ &\le |a|_0 + |\beta|_0 e^{r_0} + \sum_{j=1}^n |b_j|_0 e^{r_0} + \sum_{i=1}^m |c_i|_0 e^{r_0}|x'|_0. \end{aligned}$$

It follows that

$$r_1 = |x'|_0 \le \frac{|a|_0 + |\beta|_0 e^{r_0} + \sum_{j=1}^n |b_j|_0 e^{r_0}}{1-\sum_{i=1}^m |c_i|_0 e^{r_0}},$$

which is also contradicts (3.8). So

$$Łx \ne \lambda\mathcal{N}x + \lambda a, \lambda \in (0,1), x \in \partial\Omega.$$

Now, we define a bounded bilinear form $[.,.]$ on $C_\omega \times C_\omega^1$ by $[y,x] = \int_0^\omega y(t)x(t)dt$. Also we define $Q: y \to Coker(L)$ by $y \to \frac{1}{\omega}\int_0^\omega y(t)dt$. Obviously,

$$\{x|x \in kerL \cap \partial\Omega\} = \{x|x \equiv r_0, \text{or } x \equiv -r_0\}$$

Without loss of generality, we may assume that $x \equiv r_0$. Thus

$$\begin{aligned} & [QN(x)+Q(a),x]\cdot[QN(-x)+Q(a),x] \\ = \ & r_0^2\left[\int_0^\omega a(t)dt - e^{r_0}(\int_0^\omega \beta(t)dt + \sum_{j=1}^n \int_0^\omega b_j(t))dt\right] \\ & \left[\int_0^\omega a(t)dt - e^{-r_0}(\int_0^\omega \beta(t)dt + \sum_{j=1}^n \int_0^\omega b_j(t)dt)\right] \\ = \ & r_0^2\omega^2[\bar a - e^{r_0}(\bar\beta + \sum_{j=1}^n \bar{b_j})][\bar a - e^{-r_0}(\bar\beta + \sum_{j=1}^n \bar{b_j})]. \end{aligned} \tag{3.12}$$

As $r_0 > |\ln\frac{\bar a}{\bar\beta+\sum_{j=1}^n \bar b_j}|$, we see

$$r_0 > |\ln\frac{\bar a}{\bar\beta+\sum_{j=1}^n \bar b_j}| \ge \ln\frac{\bar a}{\bar\beta+\sum_{j=1}^n \bar b_j}$$

and

$$-r_0 < -|\ln\frac{\bar a}{\bar\beta+\sum_{j=1}^n \bar b_j}| \le \ln\frac{\bar a}{\bar\beta+\sum_{j=1}^n \bar b_j}.$$

Then

$$\bar{a} - e^{r_0}(\bar{\beta} + \sum_{j=1}^{n} \bar{b_j}) < \bar{a} - \frac{\bar{a}}{\bar{\beta} + \sum_{j=1}^{n} \bar{b}_j}(\bar{\beta} + \sum_{j=1}^{n} \bar{b_j}) = 0$$

and

$$\bar{a} - e^{-r_0}(\bar{\beta} + \sum_{j=1}^{n} \bar{b_j}) > \bar{a} - \frac{\bar{a}}{\bar{\beta} + \sum_{j=1}^{n} \bar{b}_j}(\bar{\beta} + \sum_{j=1}^{n} \bar{b_j}) = 0.$$

It follows from (3.12) that

$$[QN(x) + Q(a), x] \cdot [QN(-x) + Q(a), x] < 0.$$

Therefore, by using Lemma 2.1, we obtain that Eq.(1.2) has at least one positive $\omega-$ periodic solution.

For example, let us consider the following equation

$$N'(t) = N(t)[\sin^2 t - \frac{1}{4}N(t) - \frac{1}{4}N(t - 1 - \frac{\sin t}{2}) - \frac{1 + \frac{1}{2}\sin t}{4e^{20}}N'(t - 1 - \frac{1}{2}\cos t)]. \tag{3.13}$$

Corresponding to Eq.(1.2), we have $\omega = 2\pi, a(t) = \sin^2 t, \beta(t) = b_1(t) \equiv \frac{1}{4}$, $c_1(t) = \frac{1+\frac{1}{2}\sin t}{4e^{20}}$, $\sigma_1(t) = 1 + \frac{\sin t}{2}$ with $\sigma_1'(t) = \frac{\cos t}{2} < 1, \forall t \in [0, 2\pi]$ and $\tau_1(t) = 1 + \frac{1}{2}\cos t$ with $\tau_1'(t) = -\frac{1}{2}\sin t < 1, \forall t \in [0, 2\pi]$. So $c_{0,1}(t) = \frac{c_1(t)}{1-\tau_1'(t)} = \frac{1}{4e^{20}}$, and then $\Gamma(t) = \Gamma_1(t) = \frac{1}{4} + \frac{1}{4(1-\frac{1}{2}\cos\mu_1(t))} > 0, \forall t \in [0, 2\pi]$, where $\mu_1(t)$ is the inverse of $t - 1 - \frac{1}{2}\sin t$. So we can chose $r_0 = 20$ such that

$$\max\{\sum_{i=1}^{m} |c_i|_0, \sum_{i=1}^{m} |c_{0,i}|_0\}e^{r_0} = \max\{\frac{3}{8e^{20}}, \frac{1}{4e^{20}}\}e^{20} = \frac{3}{8} < 1.$$

Furthermore,

$$\begin{aligned}
& |\ln \frac{\bar{a}}{\bar{\beta} + \sum_{j=1}^{n} \bar{b}_j}| + \omega \frac{\overline{|a|}[1 + |\frac{\Gamma_1}{\Gamma}|_0]}{1 - \sum_{i=1}^{m} |c_{0,i}|_0 e^{r_0}} \\
= \quad & |\ln \frac{\bar{a}}{\beta + b_1}| + \omega \frac{\overline{|a|}[1 + |\frac{\Gamma_1}{\Gamma}|_0]}{1 - |c_{0,1}|_0 e^{r_0}} \\
= \quad & \frac{4\pi}{1 - \frac{1}{4}} = \frac{16\pi}{3} \\
< \quad & 20 = r_0.
\end{aligned}$$

Therefore, by applying Theorem 3.1, we see Eq.(3.13) has at least one positive $2\pi-$periodic solution.

Remark 3.1: Corresponding to [10], one can find $l_1 = \min_{m_1 \in Z} |\sigma_1 - 2m_1\pi|_0 = \frac{3}{2}$. So

$$\max\{\sum_{i=1}^{m} |c_i|_0, \sqrt{2}\sum_{j=1}^{n} |b_j|_0 l_j + \sum_{i=1}^{m} |c_{0,i}|_0\} e^{r_0} = (\frac{3\sqrt{2}}{8} + \frac{1}{4e^{20}}) e^{20} > 1,$$

which implies that condition (1.3) is not satisfied. So the result of the above example can not be obtained by [10].

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