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Riemann-Stieltjes Operators between Weighted Bloch and Weighted Bergman Spaces

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Abstract

In this paper, Riemann-Stieltjes operators between weighted Bloch and weighted Bergman spaces are considered. We characterize boundedness and compactness of these operators using certain growth properties of holomorphic symbols.

Keywords: weighted Bergman spaces, weighted Bloch spaces, Riemann-Stieltjes operator, Carleson measure

1 Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. Let $g : \mathbb{D} \to \mathbb{C}$ be a holomorphic map. Denote by $H(\mathbb{D})$ the space of holomorphic functions on $\mathbb{D}$. For $f \in H(\mathbb{D})$, the Riemann-Stieltjes operator induced by g is defined by

$$T_g f(z) = \int_0^z f(\zeta) dg(\zeta) = \int_0^1 f(tz) z g'(tz) dt, \qquad z \in \mathbb{D}.$$

The Riemann-Stieltjes operator can be viewed as a generalization of Cesaro operator defined by

$$T f(z) = \frac{1}{z} \int_0^z \frac{f(w)}{1-w} d(w), \qquad z \in \mathbb{D}.$$

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Ch. Pommerenke [**7**] initiated the study of Riemann-Stieltjes operator on H^2, where he showed that T_g is bounded on H^2 if and only if g is in $BMOA$. This was extended to other Hardy spaces H^p, $1 \leqslant p < \infty$, in [**1**]**and** [**2**] where compactness of T_g on H^p and Schatten class membership of T_g on H^2, was also completely characterized in terms of the symboy g. Similar questions on weighted Bergman spaces were considered by A. Aleman and A. G. Siskakis in [**3**]**.**

Recently, several authors have studied Riemann-Stieltjes operators on different spaces of analytic functions. For example, one can refer to ([**5**] [**8**] [**9**][**10**][**11**] and [**12**]) for the study of these operators on Bergman spaces, Dirichlet spaces, BMOA and VMOA and related references therein. In this paper we characterize boundedness and compactness of Riemann-Stieltjes operators between between weighted Bloch and weighted Bergman spaces.

2 Preliminaries

In this section we review the basic concepts of weighted Bergman spaces A^p_α and weighted Bloch spaces $\mathcal{B}^\alpha$ and collect some essential facts that will be needed throughout the paper.

2.1. Weighted Bergman Spaces. Let $dA(z)$ be the area measure on $\mathbb{D}$ normalized so that area of $\mathbb{D}$ is 1. For each $\alpha \in (-1, \infty)$, we set $d\nu_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z), z \in \mathbb{D}$. Then $d\nu_\alpha$ is a probability measure on $\mathbb{D}$. For $0 < p < \infty$ the weighted Bergman space A^p_α is defined as

$$A^p_\alpha = \{f \in H(\mathbb{D}) : ||f||_{A^p_\alpha} = \Big(\int_{\mathbb{D}} |f(z)|^p d\nu_\alpha(z)\Big)^{1/p} < \infty\}.$$

Note that $||f||_{A^p_\alpha}$ is a true norm only if $1 \leqslant p < \infty$ and in this case A^p_α is a Banach space. For $0 < p < 1$, A^p_α is a non-locally convex topological vector space and $d(f,g) = ||f-g||^p_{A^p_\alpha}$ is a complete metric for it. The growth of functions in the weighted Bergman spaces is essential in our study. To this end, the following sharp estimate will be useful. (see [**7**] p. 53.). It tells us how fast an arbitrary function from A^p_α grows near the boundary.
Let $f \in A^p_\alpha$. Then for every z in $\mathbb{D}$, we have

$$|f(z)| \leqslant \frac{||f||_{A^p_\alpha}}{(1-|z|^2)^{(2+\alpha)/p}} \tag{2.1}$$

with equality holds if and only if f is a constant multiple of the function

$$k_a(z) = \Big(\frac{1-|z|^2}{(1-\overline{a}z)^2}\Big)^{(2+\alpha)/p}. \tag{2.2}$$

It can be easily shown that $||k_a||_{A^p_\alpha} \approx 1$.

2.1. Weighted Bloch Spaces. For $\alpha > 0$, let $\mathcal{B}^\alpha$ consists of all analytic functions f on $\mathbb{D}$ satisfying the condition

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)|<\infty.$$

Note that $\mathcal{B}^1=\mathcal{B}$, the usual Bloch space . For $f\in\mathcal{B}^\alpha$ define

$$||f||_{\mathcal{B}^\alpha}=|f(0)|+\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)|<\infty.$$

With this norm $\mathcal{B}^\alpha$ is a Banach space. Integrating the estimate

$$|f'(z)|\leqslant\frac{||f||_{\mathcal{B}^\alpha}}{(1-|z|^2)^\alpha},$$

we obtain

$$|f(z)-f(0)|\leqslant\int_0^1|z||f'(tz)|dt\leqslant||f||_{\mathcal{B}^\alpha}\int_0^1\frac{|z|}{(1-t|z|^2)^\alpha}dt,$$

for all $z\in\mathbb{D}$. In case $0<\alpha<1$, the integral on the right is uniformly bounded by the constant $\int_0^1(1-t)^{-\alpha}dt$, and it follows that $\mathcal{B}^\alpha\subset H^\infty$. It is easy to check that in this case the linear space $\mathcal{B}^\alpha$ is an algebra. In fact, Hardy and Littlewood, have shown that for $0<\alpha<1$, the space $\mathcal{B}^\alpha$ consist of all functions f analytic on $\mathbb{D}$ satisfying the Lipschitz condition

$$|f(z)-f(w)|\leqslant|z-w|^{1-\alpha},$$

for all $z,w\in\mathbb{D}$ (see [**4**]).
In case $1<\alpha<\infty$, the above estimate implies

$$|f(z)-f(0)|\leqslant\frac{||f||_{\mathcal{B}^\alpha}}{\alpha-1}\frac{1}{(1-|z|^2)^{\alpha-1}},\tag{2.3}$$

while for $\alpha=1$ it is well known that the following hold ([**6**] and [**14**]).

$$|f(z)-f(w)|\leqslant||f||_{\mathcal{B}}\beta(z,w)\tag{2.4}$$

for $f\in\mathcal{B}$, where

$$\beta(z,w)=\frac{1}{2}\log\frac{|1-z\overline{w}|+|z-w|}{|1-z\overline{w}|-|z-w|}$$

is the Bergman metric on $\mathbb{D}$. From (2.4), it follows that for $f\in\mathcal{B}$,

$$|f(z)|\leqslant\frac{1}{\log 2}||f||_{\mathcal{B}}\log\Big(\frac{2}{1-|z|^2}\Big).\tag{2.5}$$

Throughout this paper we fix some positive radius $0<r<\infty$ and consider disks $D(z,r)$ in the Bergman metric. The set

$$D(z,r)=\{w\in\mathbb{D}:\beta(z,w)<r\},\qquad z\in\mathbb{D},$$

is called hyprerbolic disk or Bergman disk of radius r about z. It is well known that $D(z,r)$ is a Euclidean disk whose Euclidean center and Euclidean radius are

$$\frac{(1-s^2)z}{(1-s^2|z|^2)} \quad \text{and} \quad \frac{(1-|z|^2)s}{(1-s^2|z|^2)},$$

where $s = \tanh r \in (0,1)$, respectively. For fixed $r > 0$. the area of $D(z,r)$ in $\mathbb{D}$ has the estimation;

$$|D(z,r)|_A = \int_{D(z,r)} dA(w) \approx (1-|z|^2)^2.$$

For fixed $r > 0$, it is known that if $w \in D(z,r)$, then

$$|1 - z\overline{w}| \approx (1-|z^2|) \quad \text{and} \quad |D(w,s)|_A \approx C|D(z,r)|_A.$$

Following lemma lists additional properties of the hyperbolic disks.
Lemma 2.2.[7] Fix $r,\ 0 < r < \infty$. There exists a positive integer M and a sequence $\{a_n\}$ in $\mathbb{D}$ such that :

- (i) The disk $\mathbb{D}$ is covered by $\{D(a_n,r)\}_n$.
- (ii) Every point in $\mathbb{D}$ belongs to at most M sets in $\{D(a_n,2r)\}_n$.
- (iii) If $n \neq m$, then $\beta(a_n,a_m) \geq \frac{r}{2}$.

We shall use these estimates in the proofs of the Theorems below. For general background of weighted Bergman spaces A^p_α and weighted Bloch spaces, one may consult [**13**] and [**14**] and the references therein.

3 Riemann-Stieltjes operators from weighted Bergman spaces A^p_α into weighted Bloch spaces B^α

In this section we characterize boundedness and compactness of Riemann-Stieltjes operators from weighted Bergman spaces A^p_α into weighted Bloch spaces spaces B^α.
The following Theorem characterizes Riemann-Stieltjes operators from weighted Bergman spaces A^p_β into weighted Bloch spaces $\mathcal{B}^\alpha$.
Theorem 3.1. Let $1 \leqslant p < \infty, -1 < \beta < \infty,\ \alpha > 0$ and $g : \mathbb{D} \to \mathbb{C}$ be a holomorphic map. Then the Riemann-Stieltjes operator T_g maps A^p_β boundedly into $\mathcal{B}^\alpha$ if and only if

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{(p\alpha-\beta-2)/p}|g'(z)| < \infty. \tag{3.1}$$

Proof. First suppose that

$$M = \sup_{z\in\mathbb{D}}(1-|z|^2)^{(p\alpha-\beta-2)/p}|g'(z)| < \infty.$$

By (2.1), we have

$$|f(z)| \leqslant \frac{||f||_{A^p_\beta}}{(1-|z|^2)^{(\beta+2)/p}}$$

for all $z \in \mathbb{D}$. Thus

$$\begin{aligned} ||T_g f||_{\mathcal{B}^\alpha} &= |T_g f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|(T_g f)'(z)| \\ &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|g'(z)f(z)| \\ &\leqslant \sup_{z\in\mathbb{D}}(1-|z|^2)^{(p\alpha-\beta-2)/p}|g'(z)|\ ||f||_{A^p_\beta} \\ &= M||f||_{A^p_\beta}, \end{aligned}$$

hence T_g maps A^p_β boundedly into $\mathcal{B}^\alpha$.
Conversely, suppose T_g maps A^p_β boundedly into $\mathcal{B}^\alpha$. Fix a point a in $\mathbb{D}$ and consider the function

$$f_a(z) = \Big(\frac{1-|a|^2}{(1-\overline{a}z)^2}\Big)^{(\beta+2)/p}.$$

Then f_a is a function of unit norm in A^p_β. Since T_g maps A^p_β boundedly into $\mathcal{B}^\alpha$, we can find a positive constant C such that

$$||T_g f_a||_{\mathcal{B}^\alpha} \leqslant C||f_a||_{A^p_\beta} = C,$$

for all $a \in \mathbb{D}$, hence for each point $z \in \mathbb{D}$ we have

$$(1-|z|^2)^\alpha|f_a(z)g'(z)| \leqslant C.$$

In particular, when $z = a$, we get

$$(1-|a|^2)^\alpha\Big(\frac{1-|a|^2}{(1-|a|^2)^2}\Big)^{(\beta+2)/p}|g'(a)| \leqslant C,$$

whence

$$(1-|a|^2)^{(p\alpha-\beta-2)/p}|g'(a)| < C.$$

Since $a \in \mathbb{D}$ is arbitrary, the result follows.

Theorem 3.2. Let $1 \leqslant p < \infty, -1 < \beta < \infty,\ \alpha > 0$ and $g : \mathbb{D} \to \mathbb{C}$ be a holomorphic map. Suppose that T_g maps A^p_β boundedly into $\mathcal{B}^\alpha$. Then T_g maps A^p_β compactly into $\mathcal{B}^\alpha$ if and only if

$$\lim_{|z|\to 1^-}(1-|z|^2)^{(p\alpha-\beta-2)/p}|g'(z)| = 0. \tag{3.2}$$

Proof . First suppose that (3.2) holds. Let $\{f_n\}$ be a bounded sequence in A^p_β that converges to zero uniformly on compact subsets of $\mathbb{D}$. Let $M = \sup_n ||f_n||_{A^p_\beta} < \infty$. Given $\varepsilon > 0$, there exist an $r \in (0,1)$ such that if $|z| > r$, then

$$(1-|z|^2)^{(p\alpha-\beta-2)/p}|g'(z)| < \varepsilon.$$

Thus for $z \in \mathbb{D}$ such that $|z| > r$, by (2.1) we have

$$\begin{aligned}(1-|z|^2)^\alpha|(T_gf_n)'(z)| &= (1-|z|^2)^\alpha|g'(z)||f_n(z)| \\ &\leqslant (1-|z|^2)^{\alpha-(\beta-2)/p}|g'(z)|\ ||f_n||_{A^p_\beta} \\ &< \varepsilon M,\end{aligned}$$

for all n. On the other hand, since $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, there exist an n_0 such that if $|z| \leqslant r$ and $n \geq n_0$, then $|f_n'(z)| < \varepsilon$. By Theorem 3.1, we have $g \in \mathcal{B}^\alpha$ and so we have

$$N = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|g'(z)| < \infty,$$

hence

$$\begin{aligned}\sup_{|z|\leqslant r}(1-|z|^2)^\alpha|(T_gf_n)'(z)| &= \sup_{|z|\leqslant r}(1-|z|^2)^\alpha|g'(z)|f_n(z)| \\ &< \varepsilon N.\end{aligned}$$

The above arguments together yield

$$\begin{aligned}||T_gf_n||_{\mathcal{B}^\alpha} &= |T_gf(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|(T_g)'f_n(z)| \\ &\leqslant \sup_{|z|\leqslant r}(1-|z|^2)^\alpha|(T_gf_n)'(z)| + \sup_{|z|>r}(1-|z|^2)^\alpha|(T_gf_n)'(z)| \\ &\leqslant (N+M)\varepsilon.\end{aligned}$$

Thus

$$||T_gf_n||_{\mathcal{B}^\alpha} \to 0 \ \text{ as } \ n \to \infty,$$

hence T_g maps A^p_β compactly into $\mathcal{B}^\alpha$.

Conversely, suppose T_g maps A^p_β compactly into $\mathcal{B}^\alpha$ and (3.2) does not hold. Then there exists a positive number δ and a sequence $\{z_n\}$ in $\mathbb{D}$ such that $|z_n| \to 1$ and

$$(1-|z_n|^2)^{(p\alpha-\beta-2)/p}|g'(z_n)| \geq \delta,$$

for all n. For each n, consider the function f_n defined as

$$f_n(z) = \Big(\frac{1-|z_n|^2}{(1-\overline{z}_nz)^2}\Big)^{(\beta+2)/p}, \qquad z \in \mathbb{D}.$$

Then the sequence $\{f_n\}$ is norm bounded and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, it follows that a subsequence of $\{T_g f_n\}$ tends to 0 in $\mathcal{B}^\alpha$. On the other hand,

$$\begin{aligned}||T_g f_n||_{\mathcal{B}^\alpha} &\geq (1-|z_n|^2)^\alpha |(T_g f_n)'(z_n)| \\ &= (1-|z_n|^2)^\alpha |g'(z_n)|\ |f_n(z_n)| \\ &= (1-|z_n|^2)^{(p\alpha-\beta-2)/p} |g'(z_n)| \\ &\geq \delta,\end{aligned}$$

which is absurd. Hence we are done.

4 Riemann-Stieltjes operators between weighted Bloch spaces B^α

In this section we characterize boundedness and compactness of Riemann-Stieltjes operators between weighted Bloch spaces B^α.

Theorem 4.1. Let $\alpha > 0,\ \beta > 0$ and $g : \mathbb{D} \to \mathbb{C}$ be a holomorphic map.

(i) If $0 < \alpha < 1$, then T_g maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$ if and only if $g \in \mathcal{B}^\beta$.

(ii) Operator T_g maps $\mathcal{B}$ boundedly into $\mathcal{B}^\beta$ if and only if

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |g'(z)| \log \frac{2}{(1-|z|^2)} < \infty.$$

(iii) If $\alpha > 1$, then T_g maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$ if and only if

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^{1+\beta-\alpha} |g'(z)| < \infty.$$

Proof. First we consider the case $\alpha > 1$. Suppose T_g maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$. Consider the function

$$f_a(z) = \frac{1-|a|^2}{(1-\overline{a}z)^\alpha}, \qquad z \in \mathbb{D}.$$

Then $||f_a||_{\mathcal{B}^\beta} \leqslant 1+2\alpha$. Thus $f_a \in \mathcal{B}^\alpha$ and $M = \sup\{||f_a||_{\mathcal{B}^\beta} : a \in \mathbb{D}\} \leqslant 1+2\alpha$. Since T_g maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$, we can find a positive constant C such that

$$||T_g f_a||_{\mathcal{B}^\beta} \leqslant C||f_a||_{\mathcal{B}^\alpha} \leqslant CM$$

for each $a \in \mathbb{D}$, hence for each $z \in \mathbb{D}$, we have

$$\begin{aligned}(1-|z|^2)^\beta |g'(z)||f_a(z)| &= (1-|z|^2)^\beta |(T_g f_a)'(z)| \\ &\leqslant CM.\end{aligned}$$

In particular, when $z = a$, we have

$$(1 - |a|^2)^{1+\beta-\alpha}|g'(a)| \leqslant CM.$$

Since $a \in \mathbb{D}$ is arbitrary, the result follows.
Conversely, suppose that

$$M = \sup_{z\in\mathbb{D}}(1 - |z|^2)^{1+\beta-\alpha}|g'(z)| < \infty \tag{4.1}$$

By (2.3), we have

$$|f(z) - f(0)| \leqslant \frac{||f||_{\mathcal{B}^\alpha}}{(\alpha - 1)(1 - |z|^2)^{(\alpha-1)}}$$

for all $z \in \mathbb{D}$, independent of $f \in \mathcal{B}^\alpha$. Since

$$\begin{aligned} ||T_g f||_{\mathcal{B}^\beta} &= |T_g f(0)| + \sup_{z\in\mathbb{D}}(1 - |z|^2)^\beta|(T_g f)'(z)| \\ &\leqslant \sup_{z\in\mathbb{D}}(1 - |z|^2)^\beta|f(z) - f(0)||g'(z)| + |f(0)| \sup_{z\in\mathbb{D}}(1 - |z|^2)^\beta|g'(z)| \\ &\leqslant \sup_{z\in\mathbb{D}}(1 - |z|^2)^\beta|g'(z)| \frac{||f||_{\mathcal{B}^\beta}}{(\alpha - 1)(1 - |z|^2)^{(\alpha-1)}} + C \sup_{z\in\mathbb{D}}(1 - |z|^2)^\beta|g'(z)| \\ &\leqslant \big(CM + \frac{M}{(\alpha - 1)}\big)||f||_{\mathcal{B}^\beta}. \end{aligned}$$

Hence T_g maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$. This completes the proof of (iii). Next, we will prove (ii). Suppose T_g maps $\mathcal{B}$ boundedly into $\mathcal{B}^\beta$. For $a \in \mathbb{D}$, let

$$f_a(z) = \log \frac{2}{(1 - \overline{a}z)}, \qquad z \in \mathbb{D}.$$

Then $f_a \in \mathcal{B}$ and $||f_a||_{\mathcal{B}} \leqslant 3$. So

$$\begin{aligned} 3||T_g||_{\mathcal{B}^\beta} &\geq ||T_g f_a||_{\mathcal{B}^\beta} \\ &= |T_g f_a(0)| + \sup_{z\in\mathbb{D}}(1 - |z|^2)^\beta|(T_g f_a)'(z)| \\ &\geq (1 - |a|^2)^\beta|g'(a)||f_a(a)| \\ &= (1 - |a|^2)^\beta|g'(a)| \log \frac{2}{(1 - |a|^2)}. \end{aligned}$$

Since $a \in \mathbb{D}$ is arbitrary, the result follows.
Convesely, suppose that

$$M = \sup_{z\in\mathbb{D}}(1 - |z|^2)^\beta|g'(z)| \log \frac{2}{(1 - |z|^2)} < \infty.$$

By (2.5), for $f \in \mathcal{B}^\alpha$, we have

$$\begin{aligned} ||T_g f||_{\mathcal{B}^\beta} &= |T_g f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |(T_g f)'(z)| \\ &\leqslant \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |g'(z)| \frac{1}{\log 2} \log \frac{2}{(1-|z|^2)} ||f||_{\mathcal{B}^\alpha} \\ &= \frac{1}{\log 2} M ||f||_{\mathcal{B}}, \end{aligned}$$

hence T_g maps $\mathcal{B}$ boundedly into $\mathcal{B}^\beta$. This completes the proof of (ii). Finally we will prove (i). First suppose that T_g maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$, then

$$g = g(0) + T_g 1 \in \mathcal{B}^\beta.$$

Conversely, suppose that $g \in \mathcal{B}^\beta$. Then

$$|f(z)| \leqslant ||f||_{\mathcal{B}^\alpha}(1 + (1-|z|^2)^{1-\alpha}), \qquad \alpha \neq 1, z \in \mathbb{D}.$$

Thus, if $f \in \mathcal{B}^\alpha$, then

$$\begin{aligned} ||T_g f||_{\mathcal{B}^\beta} &\leqslant \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |(T_g f)'(z)| \\ &\leqslant \sup_{z\in\mathbb{D}}((1-|z|^2)^\beta + (1-|z|^2)^{\beta+1-\alpha})|g'(z)| \; ||f||_{\mathcal{B}^\alpha} \\ &\leqslant C \sup_{z\in\mathbb{D}}((1-|z|^2)^\beta)|g'(z)| \; ||f||_{\mathcal{B}^\alpha} \\ &\leqslant C \; ||g||_{\mathcal{B}^\beta} ||f||_{\mathcal{B}^\alpha}. \end{aligned}$$

As a result, T_g maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$.

Lemma 4.2.[**6**] Let $0 < \alpha < 1$ and let T be a bounded linear operator from $\mathcal{B}^\alpha$ into a normed linear space X. Then T is compact if and only if $||Tf_n||_X \to 0$, whenever $\{f_n\}$ is a bounded sequence in $\mathcal{B}^\alpha$ that converges to zero uniformly on $\overline{\mathbb{D}}$.

Theorem 4.3. Let $\alpha > 0$, $\beta > 0$ and $g : \mathbb{D} \to \mathbb{C}$ be a holomorphic map. Suppose that T_g maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$.

(i) If $0 < \alpha < 1$, then T_g maps $\mathcal{B}^\alpha$ compactly into $\mathcal{B}^\beta$.

(ii) Operator T_g maps $\mathcal{B}$ compactly into $\mathcal{B}^\beta$ if and only if

$$\lim_{|z|\to 1}(1-|z|^2)^\beta |g'(z)| \log \frac{2}{(1-|z|^2)} = 0.$$

(iii) If $\alpha > 1$, then T_g maps $\mathcal{B}^\alpha$ compactly into $\mathcal{B}^\beta$ if and only if

$$\lim_{|z|\to 1}(1-|z|^2)^{1+\beta-\alpha}|g'(z)| = 0.$$

Proof. First we consider the case $\alpha > 1$. To prove that the condition in (iii) is sufficient for compactness of the operator T_g from $\mathcal{B}^\alpha$ into $\mathcal{B}^\beta$, it is enough to show that if $\{f_n\}$ is a bounded sequence in $\mathcal{B}^\alpha$ that converges to zero uniformly on compact subsets of $\mathbb{D}$, then $\lim_{n\to\infty} ||T_g f_n||_{\mathcal{B}^\beta} = 0$. Let $M = \sup_n ||f_n||_{\mathcal{B}^\alpha} < \infty$. Given $\varepsilon > 0$, there exists an $r \in (0,1)$ such that, if $|z| > r$, then

$$(1-|z|^2)^{1+\beta-\alpha}|g'(z)| < \varepsilon.$$

By (2.3), we have

$$|f_n(z) - f_n(0)| \leqslant \frac{||f_n||_{\mathcal{B}^\alpha}}{(\alpha-1)(1-|z|^2)^{(\alpha-1)}}$$

for all $z \in \mathbb{D}$. Since

$$\begin{aligned}
||T_g f_n||_{\mathcal{B}^\beta} &= |T_g f_n(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|(T_g f_n)'(z)| \\
&\leqslant \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|f_n(z)-f_n(0)||g'(z)| + |f_n(0)|\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|g'(z)| \\
&\leqslant \sup_{|z|\leqslant r}(1-|z|^2)^\beta|g'(z)||f_n(z)-f_n(0)| + |f_n(0)|\sup_{|z|\leqslant r}(1-|z|^2)^\beta|g'(z)| \\
&+ \sup_{|z|>r}(1-|z|^2)^\beta|g'(z)|\;|f_n(z)-f_n(0)| + |f_n(0)|\sup_{|z|>r}(1-|z|^2)^\beta|g'(z)| \\
&\leqslant \sup_{|z|\leqslant r}(1-|z|^2)^\beta|g'(z)|\;|f_n(z)-f_n(0)| + |f_n(0)|\sup_{|z|\leqslant r}(1-|z|^2)^\beta|g'(z)| \\
&+ \sup_{|z|>r}(1-|z|^2)^\beta\frac{|g'(z)|\;\;||f||_{\mathcal{B}^\alpha}}{(\alpha-1)(1-|z|^2)^{(\alpha-1)}} + |f_n(0)|\sup_{|z|>r}(1-|z|^2)^\beta|g'(z)| \\
&\leqslant \varepsilon(4||g||_{\mathcal{B}^\beta} + \frac{M}{\alpha-1} + M) \quad \text{as} \quad n \geq n_0.
\end{aligned}$$

Thus, T_g maps $\mathcal{B}^\alpha$ compactly into $\mathcal{B}^\beta$.

Conversely, suppose that T_g maps $\mathcal{B}^\alpha$ compactly into $\mathcal{B}^\beta$ and (iii) does not hold. Then there exists a positive number δ and a sequence $\{z_n\}$ in $\mathbb{D}$ such that $|z_n| \to 1$ and

$$(1-|z_n|^2)^{1+\beta-\alpha}|g'(z_n)| \geq \delta,$$

for all n. For each n, let

$$f_n(z) = \frac{1-|z_n|^2}{(1-\overline{z}_n z)^\alpha}, \qquad z \in \mathbb{D}.$$

Then the sequence f_n is norm bounded and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. Hence there exists a subsequence of $\{T_g f_n\}$ which tends to 0 in $\mathcal{B}^\beta$. On the other hand,

$$\begin{aligned}
||T_g f_n||_{\mathcal{B}^\beta} &\geq (1-|z_n|^2)^\beta|(T_g f_n)'(z_n)| \\
&= (1-|z_n|^2)^\beta|g'(z_n)|\;|f_n(z_n)| \\
&= (1-|z_n|^2)^{1+\beta-\alpha}|g'(z_n)| \\
&\geq \delta,
\end{aligned}$$

which is absurd. Hence we are done.
Next, we will prove (ii). Let $\{f_n\}$ is a bounded sequence in $\mathcal{B}$ that converges to zero uniformly on compact subsets of $\mathbb{D}$. Let $M = \sup_n ||f_n||_{\mathcal{B}} < \infty$. Given $\varepsilon > 0$, there exists an $r \in (0,1)$ such that, if $|z| > r$, then

$$(1-|z|^2)^{\beta}|g'(z)| \log \frac{2}{(1-|z|^2)} < \varepsilon.$$

By (2.5), we have

$$|f_n(z)| \leqslant \frac{1}{\log 2} ||f_n||_{\mathcal{B}} \log \frac{2}{(1-|z|^2)}$$

for all $z \in \mathbb{D}$, $f \in \mathcal{B}^{\alpha}$. Also, by Theorem 4.1, we have $g \in \mathcal{B}^{\beta}$ and so, for the above ε, we can find $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} ||T_g f_n||_{\mathcal{B}^{\beta}} &= |T_g f_n(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)^{\beta}|g'(z)|f_n(z)| \\ &\leqslant \sup_{|z|\leqslant r} (1-|z|^2)^{\beta}|g'(z)||f_n(z)| + \sup_{|z|>r} (1-|z|^2)^{\beta}|g'(z)|\ |f_n(z)| \\ &\leqslant \varepsilon(||g||_{\mathcal{B}^{\beta}} + M) \quad \text{for} \quad n \geq n_0. \end{aligned}$$

Thus T_g maps $\mathcal{B}$ compactly into $\mathcal{B}^{\beta}$. Conversely, suppose T_g maps $\mathcal{B}$ compactly into $\mathcal{B}^{\beta}$ and condition in (ii) does not hold. Then there exists a positive number δ and a sequence $\{z_n\}$ in $\mathbb{D}$ such that $|z_n| \to 1$ and

$$(1-|z_n|^2)^{\beta}|g'(z_n)| \log \frac{2}{(1-|z_n|^2)} \geq \delta,$$

for all n. For each n, let

$$f_n(z) = \log \frac{2}{(1-\overline{z}_n z)}, \qquad z \in \mathbb{D}.$$

Then the sequence f_n is norm bounded and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. By the compactness of T_g we can find a subsequence of $\{T_g f_n\}$ which tends to 0 in $\mathcal{B}^{\beta}$. On the other hand,

$$\begin{aligned} ||T_g f_n||_{\mathcal{B}^{\beta}} &\geq (1-|z_n|^2)^{\beta}|(T_g f_n)'(z_n)| \\ &= (1-|z_n|^2)^{\beta}|g'(z_n)|\ |f_n(z_n)| \\ &= (1-|z_n|^2)^{\beta}|g'(z_n)| \log \frac{2}{(1-|z_n|^2)} \\ &\geq \delta, \end{aligned}$$

which is absurd. We are done.
Finally, we will prove (i). Suppose that $\sup_n ||f_n||_{\mathcal{B}^{\alpha}} \leqslant M$ and $f_n \to 0$ uniformly on $\overline{\mathbb{D}}$. Then

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^{\beta}|g'(z)||f_n(z)| \leqslant \sup_{|z|\leqslant 1} |f_n(z)|\ ||g||_{\mathcal{B}^{\beta}} \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

It follows from Lemma 4.2 that T_g maps $\mathcal{B}^\alpha$ compactly into $\mathcal{B}^\beta$.

Lemma 4.4. Let $1 \leqslant p < q < \infty$ and $-1 < \alpha < \infty$. Then the injection map from A^q_α into A^p_α is compact.

Proof Let $\{f_n\}_{n=1}^\infty$ be a bounded sequence in A^q_α, and let $M = \sup_{n\in\mathbb{N}} ||f_n||_{A^q_\alpha} \leqslant \infty$. By 2.1, $\{f_n : n \in \mathbb{N}\}$ is a normal family, hence we can find a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ that converges uniformly on compact subsets of $\mathbb{D}$ to an analytic function f. By Fatou's lemma,

$$\int_{\mathbb{D}} |f(z)|^p \nu_\alpha(z) \leqslant \liminf_{k\to\infty} \int_{\mathbb{D}} |f_{nk}(z)|^p \nu_\alpha(z) \leqslant M^q < \infty.$$

Since $p < q$, it follows that $f \in A^q_\alpha$. We claim that $\{f_{nk}\}$ converges to f in A^p_α. Let $\varepsilon > 0$ and let Ω be an arbitrary compact subset of $\mathbb{D}$. Now

$$\begin{aligned} \int_{\mathbb{D}\setminus\Omega} |(f_{nk} - f)(z)|^p \nu_\alpha(z) &\leqslant \Big(\int_{\mathbb{D}\setminus\Omega} |(f_{nk} - f)(z)|^q \nu_\alpha(z)\Big)^{p/q} \Big(\int_{\mathbb{D}\setminus\Omega} \nu_\alpha(z)\Big)^{1-p/q} \\ &\leqslant \Big(\int_{\mathbb{D}} |(f_{nk} - f)(z)|^q \nu_\alpha(z)\Big)^{p/q} (\nu_\alpha(\mathbb{D}\setminus\Omega))^{1-p/q} \\ &\leqslant \Big(2^q \int_{\mathbb{D}} |(f_{nk}(z)|^q + |f(z)|^q)\nu_\alpha(z)\Big)^{p/q} (\nu_\alpha(\mathbb{D}\setminus\Omega))^{1-p/q} \\ &\leqslant 2^p(||f_{nk}||^p_{A^q_\alpha} + ||f||^p_{A^q_\alpha})(\nu_\alpha(\mathbb{D}\setminus\Omega))^{1-p/q} \\ &\leqslant 2^{p+1} M^p (\nu_\alpha(\mathbb{D}\setminus\Omega))^{1-p/q}, \end{aligned}$$

where in the first line we have used Holder's inequality and the elementary inequalities

$$(x+y)^a \leqslant 2^a(x^a + y^b), \quad (x+y)^b \leqslant (x^b + y^b)$$

which holds when $x, y \geq 0$, and $0 < b < 1$. By choosing the compact set Ω so that $\mathbb{D}\setminus\Omega$ has sufficiently small area, we obtain

$$\int_{\mathbb{D}\setminus\Omega} |(f_{nk} - f)(z)|^p \nu_\alpha(z) < \varepsilon/2$$

for k large enough. On the other hand, since $f_n \to f$ uniformly on Ω we can choose k large enough so that

$$\int_{\Omega} |(f_{nk} - f)(z)|^p \nu_\alpha(z) < \varepsilon/2.$$

Thus $\{f_{nk}\}$ converges to f in A^p_α. Hence the injection map is compact.

Corollary 4.5. Let $1 \leqslant p < \infty$, $-1 < \beta < \infty$, $\alpha > 0$ and $g : \mathbb{D} \to \mathbb{C}$ be holomorphic. Then T_g maps $\mathcal{B}^\alpha$ compactly into A^p_β if and only if T_g maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$.

Proof. Suppose T_g maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$, thus also into the large space

A^p_β. Since convergence in either space implies uniform convergence on compact sets, it follows from closed graph theorem that T_g maps $\mathcal{B}^\alpha$ boundedly into A^p_β. In order to show that T_g maps $\mathcal{B}^\alpha$ compactly into A^p_β, choose any q such that $q > p$ and factorize T_g through the intermediate space A^q_β :

$$\mathcal{B}^\alpha \xrightarrow{\widetilde{T}_g} A^q_\beta \xrightarrow{I} A^p_\beta,$$

where $\widetilde{T}_g$ is the Riemann-Stieltjes operator from $\mathcal{B}^\alpha$ to A^q_β, and I is the injection map. Since I is compact and $\widetilde{T}_g$ is bounded, so T_g maps $\mathcal{B}^\alpha$ compactly into A^p_β.

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