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A Result on Diffuse Random Measure

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Abstract

We prove a result for a diffuse random measure with Radon intensity, from which it follows that if ϕ and ψ are two random measures in Γ defined on $(\mathfrak{M}^{(d)}, \mathcal{M}^{(d)}, P)$ such that
E($\phi(I) \mid \mathcal{M}^{(d)}_{I^c}$)= E($\psi(I) \mid \mathcal{M}^{(d)}_{I^c}$) P $-$ *a.s.*
for all I in the union of a system of a partitions of Γ , then $\phi = \psi$ P-a.s.

Keywords: random measure, diffuse, intensity, conditional, Radon

1 Introduction

Let Γ be a locally compact second countable Hausdorff topological space, i.e. a polish space, $\mathfrak{B}$ the σ -algebra of Borel subsets of Γ , $\mathfrak{B}_0$ the family of bounded sets in $\mathfrak{B}$ and let $\mathfrak{M}$ be the space of non negative Borel measures which are finite on $\mathfrak{B}_0$, the space of Radon Measures, and $\mathcal{M}$ be the σ - algebra in $\mathfrak{M}$ generated by the mappings $\mu \to \mu(B)$, $B \in \mathfrak{B}_0$, (i.e. smallest σ - algebra making these mappings measurable) and $\mathfrak{M}^{(d)}$ denotes the subset of $\mathfrak{M}$ consisting of all diffuse measures and $\mathcal{M}^{(d)} = \mathcal{M} \bigcap \mathfrak{M}^{(d)}$. For $I \in \mathfrak{B}$, $\mathcal{M}_I$ denotes the σ - algebra in $\mathfrak{M}$ generated by mappings $\mu \to \mu(B)$, $B \in \mathfrak{B}_0$, $B \subseteq I$. $\mathcal{M}^{(d)}_I$ is defined similarly for any $I \in \mathfrak{B}$. *A* random measure in Γ is a probability measure on $(\mathfrak{M}, \mathcal{M})$ or a measurable function from some probability space into $(\mathfrak{M}, \mathcal{M})$. Diffuse random measure is defined similarly. If ξ is a random measure, it is clear that for fixed $I \in \mathfrak{B}_0$, the mapping $\omega \to \xi(\omega, I)$ is a random variable, which will be denoted by $\xi(I)$. Also $\xi^{(d)}(I)$ denotes the restriction of $\xi(I)$ on $\mathfrak{M}^{(d)}$ .For a Radon measure ω in Γ and any $I \in \mathfrak{B}$, define $\bar{\xi}(\omega, I) = \omega(I)$. If P is a probability measure on $(\mathfrak{M}, \mathcal{M})$, the measure in Γ given by E($\bar{\xi}(I)$)= $\int \bar{\xi}(\omega, I)P(d\omega)$,I $\in \mathfrak{B}$, is called the intensity of P. Similarly if ξ is a measurable function from some probability space into

$(\mathfrak{M}, \mathcal{M})$, the measure in Γ given by E($\xi(I)$) is called the intensity of ξ . Finally a system of partitions of Γ (compare with null array of partitions in [3] and see [1]) means a sequence $\{\Delta_n\}_{n\geq 1}$ of partitions of Γ having the following properties:

1. Each Δ_n consists of countable many bounded Borel sets.
2. Δ_{n+1} is a refinement of Δ_n for all $n \geq 1$.
3. For any $\Gamma' \subseteq \Gamma$, $\Gamma' \in \mathfrak{B}_0$, and any $\varepsilon > 0$, there is some $n \geq 1$, such that Γ' can be covered by a finite number of elements of Δ_n each having diameter smaller than ε .

The topological structure of Γ ensures the existence of such a sequence of partitions (see [2]).

In this article we prove that if ϕ and ψ are two random measure in Γ defined on

$(\mathfrak{M}^{(d)}$,$\mathcal{M}^{(d)}$,P), with Radon intensity, such that

$$\text{E}(\ \phi(I) \mid \mathcal{M}^{(d)}_{I^c}\ )=\text{E}(\ \psi(I) \mid \mathcal{M}^{(d)}_{I^c}\ )\ \text{P} - a.s. \qquad (1)$$

for all $I \in \bigcup\limits_{n\geq 1} \Delta_n$, where I^c represents the complement of I , then $\phi = \psi$ P- a.s. (partially case was done in [5]).

2 Main Theorem

We first represent the following lemmas.

Lemma 1 . If ϕ and ψ are two random measures in Γ on $(\mathfrak{M}^{(d)}, \mathcal{M}^{(d)}, P)$ and satisfying (1) , for any $I \in \bigcup\limits_{n\geq 1} \Delta_n$ and disjoint $I_1, I_2, ..., I_r \in \bigcup\limits_{n\geq 1} \Delta_n$ and any $a_1, a_2, ..., a_r \geq 0$:

$$\int_H \phi(\omega, I)P(d\omega) = \int_H \psi(\omega, I)P(d\omega) \qquad (2)$$

Where $H=\{\ \bar{\xi}(I_1) \leq a_1\ , \ldots ,\ \bar{\xi}(I_r) \leq a_r\ \}$.

Lemma 2 . Two finite measures μ_1 and μ_2 on $(\mathfrak{M}^{(d)}$, $\mathcal{M}^{(d)}$) with μ_1 ($\mathfrak{M}^{(d)}$)= μ_2 ($\mathfrak{M}^{(d)}$) are identical if, and only if, for any disjoint $I_1, I_2, ..., I_r \in \bigcup\limits_{n\geq 1} \Delta_n$, ($\bar{\xi}(I_1)_1$, . . . , $\bar{\xi}(I_r)$) maps μ_1 and μ_2 to the same measure in $[\ 0, +\infty\ )^{\ r}$.

Theorem

If ϕ and ψ are two random measures in Γ defined on $(\mathfrak{M}^{(d)}$, $\mathcal{M}^{(d)}$,P), ϕ, ψ have Radon intensity,

such that $$\text{E}(\ \phi(I) \mid \mathcal{M}^{(d)}_{I^c}\ )=\text{E}(\ \psi(I) \mid \mathcal{M}^{(d)}_{I^c}\ )\ \text{P} - a.s.$$

for all I $\in \bigcup\limits_{n\geq 1} \Delta_n$, then $\phi = \psi$ P- a.s.

Proof.

For fixed I $\in \bigcup_{n\geq 1} \Delta_n$, define a measure on $\mathcal{M}^{(d)}$ by

$\mu_1(A) = \int_A \phi(\omega, I)P(d\omega)$, $A \in \mathcal{M}^{(d)}$,

and one by

$\mu_2(A) = \int_A \psi(\omega, I)P(d\omega)$, $A \in \mathcal{M}^{(d)}$.

Relation (2), in view of lemma 2 , clearly implies that $\mu_1 = \mu_2$, i.e.

$$\int_A \phi(\omega, I)P(d\omega) = \int_A \psi(\omega, I)P(d\omega)$$

for all $A \in \mathcal{M}^{(d)}$. This in turn implies that for P- *a.e.* ω,

$$\phi(\omega, I) = \psi(\omega, I).$$

Since $\bigcup_{n\geq 1} \Delta_n$ is countable, it follows that there is a P-null event N such that if $\omega \notin N$, then

$$\phi(\omega, S) = \psi(\omega, S)$$

for all $S \in \bigcup_{n\geq 1} \Delta_n$ and therefore $S \in$ B .

The proof of theorem is thus complete.

3 Proof of the lemmas

Proof of lemma 1:

Suppose I , $I_0 \in \bigcup_{n\geq 1} \Delta_n$ with $I \subseteq I_0$, and $a \geq 0$, $A \in \mathcal{M}^{(d)}_{I_0^c}$, are fixed. Define, for $n \geq \nu(I)$,($\nu(I)$ denotes the smallest positive integer n such that I can be expressed as a union of elements of Δ_n)

$$X_n(\omega) = \sum_{J \in \Delta_n^{(I)}} \chi_{\{\bar{\xi}(I_0 \setminus J) \leq a\}}(\omega)\phi(\omega, J).$$

Where $\Delta_n^{(I)}$ denotes the family $\{ J \in \Delta_n : J \subseteq I \}$.

For each $\omega \in \mathfrak{M}^{(d)}$, on account of its diffuseness,

$$\lim_{n\to\infty} \max_{J \in \Delta_n^{(I)}} \bar{\xi}(\omega, J) = 0.$$

Therefore

$$\lim_{n\to\infty} X_n(\omega) = \chi_{\{\bar{\xi}(I_0)\le a\}}(\omega)\phi(\omega, I).$$

Since $X_n(\omega) \le \phi(\omega, I)$, from the dominated convergence theorem it follows that

$$\lim_{n\to\infty} E(\chi_A X_n) = E(\chi_{A\bigcap\{\bar{\xi}(I_0)\le a\}}\phi(I)).$$

If we define, for $n \ge \nu(I)$,

$$Y_n(\omega) = \sum_{J\in\Delta_n^{(I)}} \chi_{\{\bar{\xi}(I_0\setminus J)\le a\}}(\omega)\psi(\omega, J),$$

It follows similarly that

$$\lim_{n\to\infty} E(\chi_A Y_n) = E(\chi_{A\bigcap\{\bar{\xi}(I_0)\le a\}}\psi(I)).$$

For each $n \ge \nu(I)$, in view of (1) we have

$$E(\chi_A X_n) = E(\chi_A Y_n).$$

It therefore follows that

$$E(\chi_{A\bigcap\{\bar{\xi}(I_0)\le a\}}\phi(I)) = E(\chi_{A\bigcap\{\bar{\xi}(I_0)\le a\}}\psi(I)). \tag{3}$$

Now, if $I_1, I_2, ..., I_r \in \bigcup_{n\ge1} \Delta_n$ are disjoint, it is not difficult to see that we can choose n_0 such that for each $J \in \Delta_{n_0}^{(I)}$, either $J \subseteq I_s$ for some s, $1 \le s \le r$, or $J \subseteq \Gamma\setminus(I_1 \bigcup ... \bigcup I_r)$.

Then

$$\int_H \phi(\omega, I)P(d\omega)$$

$$= \sum_{J\in\Delta_{n_0}^{(I)}, J\subseteq\bigcup I_s} \int_H \phi(\omega, J)P(d\omega) + \sum_{J\in\Delta_{n_0}^{(I)}, J\subseteq\Gamma\setminus(\bigcup I_s)} \int_H \phi(\omega, J)P(d\omega)$$

$$= \sum_{J \in \Delta_{n_0}^{(I)}, J \subseteq \bigcup I_s} \int_H \psi(\omega, J) P(d\omega) + \sum_{J \in \Delta_{n_0}^{(I)}, J \subseteq \Gamma \setminus (\bigcup I_s)} \int_H \psi(\omega, J) P(d\omega)$$

from (3) and (1), respectively,

$$= \int_H \psi(\omega, I) P(d\omega).$$

Proof of lemma 2:

This lemma is a consequence of the following lemma which is consequence of lemma I.2.1 and Theorem 3.1 of [3].

Lemma 3:

Let ϕ and ψ be two random measures in Γ (two measurable functions from some probability space into $(\mathfrak{M}, \mathcal{M})$). ϕ and ψ have the same distribution in $\mathfrak{M}$ if, and only if, for any disjoint $I_1, I_2, ..., I_r \in \bigcup_{n \geq 1} \Delta_n$, the random vectors $(\phi(I_1), ..., \phi(I_r))$ and $(\psi(I_1), ..., \psi(I_r))$ are identically distributed .

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