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Monotonicity of Some Functions Involving the Psi Function and Sharp Estimates for the Remainders of Certain Series

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Abstract

Let $s, t > 0, t > s$ and $t - s \neq 1$. We prove that the function $\frac{t-s}{\psi(x+t)-\psi(x+s)} - x$ is decreasing on $(-s, \infty)$ if $t-s < 1$ and increasing on the same interval if $t - s > 1$. We apply this result in finding sharp estimates for the error term of several series such as $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k-1+a}$, $a > 0$. Our results generalize a recent theorem of L. Tóth and J. Bukor.

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1 Introduction and results

Let $\sum_{k=1}^{\infty}(-1)^{k-1}a_k$ be a Leibniz series, i.e., $0 < a_{k+1} < a_k$, for every $k \geq 1$ and $\lim_{k\to\infty} a_k = 0$. Over the years, the problem of finding sharp estimates for the error term $\left|\sum_{k=n+1}^{\infty}(-1)^{k-1}a_k\right|$, where a_k is a specific sequence, has been studied by several authors. In a recent paper [10], L. Tóth and J. Bukor considered the case where $a_k = \frac{1}{k}$ and showed that the inequalities

$$\frac{1}{2n+a} \leq \Big| \sum_{k=n+1}^{\infty} (-1)^{k-1} \frac{1}{k} \Big| < \frac{1}{2n+b}\,, \tag{1.1}$$

hold for every $n \geq 1$, where $a = \frac{1}{1-\log 2} - 2$ and $b = 1$ are the best constants (the smallest a and the largest b). The same question can be raised for other special cases of a sequence a_k such as for example $a_k = \frac{1}{2k-1}$ (cf. [9]).

In a very interesting paper, V. Timofte [7], proved a general result in the case where $a_k = f(k)$ and f is a continuous convex function, which enabled him to find estimates for the error $\big| \sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{k^\alpha} \big|$, $\alpha > 0$, and obtain the result of [10] as the particular case $\alpha = 1$.

In the spirit of [7], [9], [10], we seek sharp estimates of type (1.1), for the error $\big| \sum_{k=n+1}^{\infty} (-1)^{k-1} \frac{1}{k-1+a} \big|$, $a > 0$. Our results generalize also the main Theorem of [10] and sharpen a result of [9]. These estimates can also be obtained by the method of [7], [8]. Details regarding this approach and other results of this type will appear elsewhere [4]. Here we follow a totally different approach from [7] and [10]. More specifically, our sharp inequalities follow easily from the monotonicity of a function involving the ψ function defined by $\psi(x) = \dfrac{\Gamma'(x)}{\Gamma(x)}$, where $\Gamma(x)$ is Euler's gamma function. A result of N. Elezović, C. Giordano and J. Pečarić [3] plays a key role in our proof.

Over the past half century several properties and inequalities for the ψ and related functions with many applications in different areas of Mathematics have been proved by several authors. See the recent article [6] and the list of references therein. In this paper, we modestly add few more applications of these very important functions.

The main result we prove here, is the following.

Theorem 1. *Let s, $t > 0$ and $t > s$, $t - s \neq 1$. Then the function*

$$K(x) := \frac{t-s}{\psi(x+t) - \psi(x+s)} - x$$

is strictly decreasing on $(-s, \infty)$ for $t - s < 1$, and strictly increasing on the same interval for $t - s > 1$.

Note that the function $K(x)$ is identically equal to s for $t - s = 1$.

Using the asymptotic formula

$$\frac{\psi(x+t)-\psi(x+s)}{t-s}=\frac{1}{x}+\frac{1-s-t}{2\,x^2}+O\Big(\frac{1}{x^3}\Big),\quad \text{as } x\to\infty\,,$$

(see, [2]), we easily infer that

$$\lim_{x\to\infty} K(x)=\frac{s+t-1}{2}\,.$$

This and Theorem 1 yield

Corollary 1. *Let $s,\, t>0,\ t>s,\ t-s\neq 1$ and $x_0\in(-s,\,\infty)$. Then for all $x\geq x_0$ we have*

$$\frac{t-s}{x+K(x_0)}\leq \psi(x+t)-\psi(x+s)<\frac{2\,(t-s)}{2x+s+t-1} \tag{1.2}$$

when $t-s<1$. The reversed inequalities are valid when $t-s>1$. The bounds are the best possible.

Note that the second inequality in (1.2) follows also by a combination of the inequalities (7) and (9) of [3] and compare paper [1] for some special cases of (1.2). Various inequalities for differences of psi functions can be found in [6].

Using inequalities (1.2) we obtain the following result:

Theorem 2. *Let $a>0$. For all positive integers n, the inequalities*

$$\frac{1}{2n+A}\leq\Big|\sum_{k=n+1}^{\infty}(-1)^{k-1}\frac{1}{k-1+a}\Big|<\frac{1}{2n+B}\,, \tag{1.3}$$

hold, where

$$A=\frac{2}{\psi\big(\frac{a+2}{2}\big)-\psi\big(\frac{a+1}{2}\big)}-2,\quad \textit{and}\ \ B=2a-1$$

are the best constants.

This generalizes the results of [9] and [10].

In a similar manner we obtain a relevant result:

Theorem 3. *Let $a,b>0,\ b>a$. If $b-a<1$, then for all positive integers n, the inequalities*

$$\frac{1}{n+C}\leq\sum_{k=n+1}^{\infty}\frac{1}{(k+a)(k+b)}<\frac{1}{n+D}\,, \tag{1.4}$$

hold, with best constants

$$C = \frac{b-a}{\psi(b+2)-\psi(a+2)} - 1, \quad and \quad D = \frac{a+b+1}{2}\,.$$

The reversed inequalities are valid when $b-a>1$.

In the next section we give the proofs of the results stated above. In the final section we provide some numerical examples of the estimates (1.3) and (1.4).

2 Proofs of the results

We first prove Theorem 1. Let

$$D(x) := \psi(x+t) - \psi(x+s) \ \text{ and } \ \phi(x) := \Big(\frac{\Gamma(x+t)}{\Gamma(x+s)}\Big)^{\frac{1}{t-s}} - x.$$

We observe that

$$\Big(\frac{\Gamma(x+t)}{\Gamma(x+s)}\Big)' = \frac{\Gamma(x+t)}{\Gamma(x+s)}\, D(x)\,.$$

Hence

$$1+\phi'(x) = \frac{D(x)}{t-s}\Big(\frac{\Gamma(x+t)}{\Gamma(x+s)}\Big)^{\frac{1}{t-s}} = \frac{D(x)}{t-s}\,(x+\phi(x))\,.$$

It follows from this that

$$D(x) = \frac{(t-s)(1+\phi'(x))}{x+\phi(x)},$$

so that

$$K(x) = \frac{\phi(x) - x\,\phi'(x)}{1+\phi'(x)}\,.$$

Differentiating we obtain

$$K'(x) = -\frac{\phi''(x)}{(1+\phi'(x))^2}\Big(\frac{\Gamma(x+t)}{\Gamma(x+s)}\Big)^{\frac{1}{t-s}}\,. \tag{2.1}$$

It has been shown in [3] that the function $\phi(x)$ is convex on $(-s,\infty)$ when $0<t-s<1$, and concave on the same interval when $t-s>1$. This in combination with (2.1) concludes the proof of Theorem 1. □

For the proof of Theorem 2 we observe that

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k-1+a} = \frac{1}{2}\Big[\psi\Big(\frac{a+1}{2}\Big) - \psi\Big(\frac{a}{2}\Big)\Big]$$

and

$$\sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k-1+a} = \frac{1}{2}\left[\psi\Big(\frac{a+1}{2}\Big)-\psi\Big(\frac{a}{2}\Big)\right]$$
$$(-1)^{n+1}\frac{1}{2}\left[\psi\Big(\frac{n+a+1}{2}\Big)-\psi\Big(\frac{n+a}{2}\Big)\right], \quad n=1,2,\dots,$$

see [5]. An application of (1.2) yields (1.3) and completes the proof of Theorem 2. □

In order to prove Theorem 3 we let $b > a > 0$. Then using the formulas

$$\sum_{k=0}^{\infty}\frac{1}{(k+a)(k+b)} = \frac{\psi(b)-\psi(a)}{b-a},$$

and

$$\sum_{k=0}^{n}\frac{1}{(k+a)(k+b)} = \frac{\psi(b)-\psi(a)+\psi(n+a+1)-\psi(n+b+1)}{b-a}, \; n=1,2,\dots$$

(see [5]) together with Corollary 1, we conclude the proof of Theorem 3. □

3 Applications

In this Section we apply Theorems 2 and 3 in order to find sharp estimates of the error terms of several numerical series. Let us consider first the classical series

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k} = \log 2, \quad \sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{2k-1} = \frac{\pi}{4}.$$

According to Theorem 2, the best constants c_1, c_2, such that the inequalities

$$\frac{1}{2n+c_1} \le \Big|\sum_{k=n+1}^{\infty}(-1)^{k-1}\frac{1}{k}\Big| < \frac{1}{2n+c_2} \tag{3.1}$$

hold for every $n \ge 1$, are $c_1 = \dfrac{2}{\psi(3/2)-\psi(1)} - 2 = \dfrac{1}{1-\log 2} - 2$ and $c_2 = 1$. This is the main result of [10].

Similarly, the inequalities

$$\frac{1}{4n+c_3} \le \Big|\sum_{k=n+1}^{\infty}(-1)^{k-1}\frac{1}{2k-1}\Big| < \frac{1}{4n+c_4}, \tag{3.2}$$

hold for every $n \geq 1$, with best constants $c_3 = \dfrac{4}{\psi(5/4) - \psi(3/4)} - 4 = \dfrac{4}{4-\pi} - 4$, and $c_4 = 0$. The question regarding the best constants c_3 and c_4 raised also in [10] without a specific answer. In [9], the author gave the estimates (3.2) with constants $c = 2\sqrt{19} - 8 = 0.7177\ldots > c_3 = 0.65979\ldots$ and $d = 0 = c_4$.

Next we give some other examples of this type. Let us consider the series

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{3k-2} = \frac{1}{3}\left(\frac{\pi\sqrt{3}}{3} + \log 2\right)$$

and

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{3k-1} = \frac{1}{3}\left(\frac{\pi\sqrt{3}}{3} - \log 2\right).$$

Then for all $n \geq 1$, we have

$$\frac{1}{6n + c_5} \leq \Big|\sum_{k=n+1}^{\infty}(-1)^{k-1}\frac{1}{3k-2}\Big| < \frac{1}{6n + c_6},$$

with best constants $c_5 = \dfrac{3}{3 - \log 2 - \frac{\pi\sqrt{3}}{3}} - 6 = 0.08453\ldots$ and $c_6 = -1$ and

$$\frac{1}{6n + c_7} \leq \Big|\sum_{k=n+1}^{\infty}(-1)^{k-1}\frac{1}{3k-1}\Big| < \frac{1}{6n + c_8},$$

with best constants $c_7 = \dfrac{6}{3 + 2\log 2 - \frac{2\pi\sqrt{3}}{3}} - 6 = 1.9083\ldots$ and $c_8 = 1$.

Some other interesting series are

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{4k-1} = \frac{\sqrt{2}}{8}\left(\pi - 2\log(1+\sqrt{2})\right)$$

and

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{4k-3} = \frac{\sqrt{2}}{8}\left(\pi + 2\log(1+\sqrt{2})\right).$$

Then for all $n \geq 1$, we have

$$\frac{1}{8n + c_9} \leq \Big|\sum_{k=n+1}^{\infty}(-1)^{k-1}\frac{1}{4k-1}\Big| < \frac{1}{8n + c_{10}},$$

with best constants $c_9 = \dfrac{24}{8 - 3\pi\sqrt{2} + 6\sqrt{2}\,\log(1+\sqrt{2})} - 8 = 3.1625\ldots$ and $c_{10} = 2$, while

$$\frac{1}{8n+c_{11}} \leq \Big|\sum_{k=n+1}^{\infty}(-1)^{k-1}\frac{1}{4k-3}\Big| < \frac{1}{8n+c_{12}}\,,$$

with best constants $c_{11} = \dfrac{8}{8 - \sqrt{2}(\pi + 2\,\log(1+\sqrt{2}))} - 8 = -0.48272\ldots$ and $c_{12} = -2$.

Numerous other examples of sharp inequalities of this type can be derived using Theorem 3.

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