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Increasing and Decreasing Gaps: Numbers which Have just h Prime Factors

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Abstract

Let us consider the numbers which have just h primes in their factorization. The sequence of these numbers will be denoted by $c_{n,h}$. We prove that the inequalities

$$c_{n+1,h} - c_{n,h} < c_{n,h} - c_{n-1,h}$$

$$c_{n+1,h} - c_{n,h} > c_{n,h} - c_{n-1,h}$$

both have infinitely many solutions.

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1 Introduction

Let the inequalities

$$\frac{p_{n-1} + p_{n+1}}{2} < p_n \qquad \frac{p_{n-1} + p_{n+1}}{2} > p_n$$

or

$$p_{n+1} - p_n < p_n - p_{n-1} \tag{1}$$

$$p_{n+1} - p_n > p_n - p_{n-1} \tag{2}$$

It was proved by Erdos and Turan [1] that the inequalities (1) and (2) both have infinitely many solutions (p_n is the n-th prime). Two proofs were given, an elementary one and an analytical proof. Let us consider the numbers which have just h primes in their factorization. The sequence of these numbers will be denoted by $c_{n,h}$. First, we prove that inequalities (1) and (2) both have infinitely many solutions using a new and very elementary method. Then, the method is applied to prove

$$c_{n+1,h} - c_{n,h} < c_{n,h} - c_{n-1,h} \tag{3}$$

$$c_{n+1,h} - c_{n,h} > c_{n,h} - c_{n-1,h} \tag{4}$$

have infinitely many solutions.

2 An elementary proof that the inequalities (1) and (2) both have infinitely many solutions

Lemma 2.1 *If*

$$p_{k+1} = p_k + d, \qquad p_{k+2} = p_k + 2d, \ldots, p_{k+s} = p_k + sd \qquad (s \geq 1)$$

Then $s \leq d$

Proof. We can write

$$p_k = (d+1)h + r \qquad (0 \leq r \leq d)$$

If $r > 0$, then

$$p_k + rd = (d+1)(h+r)$$

is a composite number. Then $s \leq d-1$.
If $r = 0$ we have $p_k = d+1$.
If $d \geq 4$ then

$$p_k + (d-3)d = (d-1)^2$$

is a composite number. Hence $s \leq d-4$.
If $d = 2$ then $p_2 = 3$, $p_3 = 5$, $p_4 = 7$ and 9 is composite.Therefore $s = 2$. The lemma is proved.

The following theorem (Euler) is well known and it can be proved by elementary methods. We use it as a lemma.

Lemma 2.2 *The series* $\sum_{i=1}^{\infty} \frac{1}{p_i}$ *is divergent.*

Theorem 2.3 *The inequalities (1) and (2) both have infinitely many solutions.*

Proof. Consider the inequality (1).

Suppose that $k \geq n+1$ we have

$$p_k - p_{k-1} \leq p_{k+1} - p_k$$

Let $p_{n+1} - p_n = a$.

Then, we have the following inequalities (see lemma (2.1))

$$p_n = q_n$$

$$p_{n+1} \geq q_{n+1} = p_n + a$$
$$p_{n+2} \geq q_{n+2} = p_n + 2a$$
$$\vdots$$
$$p_{n+a} \geq q_{n+a} = p_n + a^2$$
$$p_{n+a+1} \geq q_{n+a+1} = p_n + a^2 + (a+1)$$
$$p_{n+a+2} \geq q_{n+a+2} = p_n + a^2 + 2(a+1)$$
$$\vdots$$
$$p_{n+a+(a+1)} \geq q_{n+a+(a+1)} = p_n + a^2 + (a+1)^2$$
$$\vdots$$

$$\begin{aligned} p_{n+a+(a+1)+\ldots+(a+k)} &\geq q_{n+a+(a+1)+\ldots+(a+k)} \\ &= p_n + a^2 + (a+1)^2 + \ldots + (a+k)^2 \\ &= (1/3)k^3 + (a+1/2)k^2 + (a^2+a+1/6)k + p_n + a^2 \end{aligned}$$

$$\vdots$$

Therefore

$$\begin{aligned} \sum_{i=n}^{\infty} \frac{1}{p_i} &\leq \sum_{i=n}^{\infty} \frac{1}{q_i} \leq \frac{a}{q_n} + \sum_{k=0}^{\infty} \frac{a+k+1}{q_{n+a+(a+1)+\ldots+(a+k)}} \\ &= \frac{a}{p_n} + \sum_{k=0}^{\infty} \frac{k+a+1}{(1/3)k^3 + (a+1/2)k^2 + (a^2+a+1/6)k + p_n + a^2} \end{aligned}$$

Thus, the series $\sum_{i=n}^{\infty} \frac{1}{p_i}$ is convergent, which contradicts lemma (2.2). Therefore the inequality (1) has infinitely many solutions.

On the other hand, if the inequality (1) has infinitely many solutions then clearly the inequality (2) also has infinitely many solutions. The theorem is proved.

Note that this proof is more elementary than the Erdos-Turan proofs [1] since we do not need none assumption on the distribution of prime numbers.

3 Main Results

We need the following lemmas.

Lemma 3.1 *The series* $\sum_{i=1}^{\infty} \frac{1}{c_{i,h}}$ *is divergent.*

Proof. We have

$$\sum_{i=1}^{\infty} \frac{1}{c_{i,h}} \geq \sum_{i=1}^{\infty} \frac{1}{2^{h-1} p_i}$$

The second series is divergent (lemma 2.2), consequently also the first series is divergent. The lemma is proved.

The following lemma is well known

Lemma 3.2 *For* $k \geq 1$ *and* $m \geq 1$ *we have* $H_m(k) = 1^m + \ldots + k^m$ *is a polynomial in* k *of degree* $m+1$*with rational coefficients.*

For example, we have

$$H_1(k) = \frac{1}{2}k^2 + \frac{1}{2}k \qquad H_2(k) = \frac{1}{3}k^3 + \frac{1}{2}k^2 + \frac{1}{6}k$$

Lemma 3.3 *given the equalities*

$$c_{k+1,h} = c_{k,h} + d, \qquad c_{k+2,h} = c_{k,h} + 2d, \ldots, c_{k+s,h} = c_{k,h} + sd \qquad (s \geq 1)$$

then if d *is big enough* $(h \geq 2)$

$$s \leq d^{2h} = P(d) \tag{5}$$

Proof. We can write

$$c_{k,h} = (d+1)^h t + r, \qquad (0 \leq r \leq (d+1)^h - 1)$$

If $r > 0$, then the number

$$c_{k,h} + r\left(\frac{(d+1)^h - 1}{d}\right) d = (d+1)^h (t+r)$$

has at least $h+1$ prime factors. Hence

$$s \leq \frac{((d+1)^h - 1)^2}{d} \tag{6}$$

where the right hand side is a polynomial in d of degree $2h-1$. If $r=0$, then $t=1$, and the number

$$c_{k,h} + (d+1)^h d = (d+1)^{h+1}$$

has at least $h+1$ prime factors. Therefore

$$s \leq (d+1)^h \tag{7}$$

where the right hand side is a polynomial in d of degree h. Finally, if d is big enough (6) and (7) give (5).The lemma is proved.

Theorem 3.4 *Inequalities (3) and (4) have infinitely many solutions.*

Proof. Let us consider the inequality (3). Suppose that $k \geq n+1$ we have

$$c_{k,h} - c_{k-1,h} \leq c_{k+1,h} - c_{k,h}$$

Let $c_{n+1,h} - c_{n,h} = a$.
Then, we have the following inequalities (see lemma (3.3))

$$c_{n,h} = q_n$$

$$c_{n+1,h} \geq q_{n+1} = c_{n,h} + a$$
$$c_{n+2,h} \geq q_{n+2} = c_{n,h} + 2a$$
$$\vdots$$
$$c_{n+P(a),h} \geq q_{n+P(a)} = c_{n,h} + P(a)a$$
$$c_{n+P(a)+1,h} \geq q_{n+P(a)+1} = c_{n,h} + P(a)a + (a+1)$$
$$c_{n+P(a)+2,h} \geq q_{n+P(a)+2} = c_{n,h} + P(a)a + 2(a+1)$$
$$\vdots$$
$$c_{n+P(a)+P(a+1),h} \geq q_{n+P(a)+P(a+1)} = c_{n,h} + P(a)a + P(a+1)(a+1)$$
$$\vdots$$

$$\begin{aligned} c_{n+P(a)+P(a+1)+\ldots+P(a+k),h} &\geq q_{n+P(a)+P(a+1)+\ldots+P(a+k)} \\ &= c_{n,h} + P(a)a + P(a+1)(a+1) + \ldots \\ &+ P(a+k)(a+k) = Q(k) \end{aligned}$$

$$\vdots$$

Where $Q(k)$ (lemma (3.3) and lemma (3.2)) is a polynomial in k of degree $2h+2$. Therefore

$$\sum_{i=n}^{\infty} \frac{1}{c_{i,h}} \leq \sum_{i=n}^{\infty} \frac{1}{q_i} \leq \frac{P(a)}{q_n} + \sum_{k=0}^{\infty} \frac{P(a+k+1)}{q_{n+P(a)+P(a+1)+\ldots+P(a+k)}} = \frac{P(a)}{c_{n,h}} + \sum_{k=0}^{\infty} \frac{P(a+k+1)}{Q(k)}$$

Where $P(a+k+1)$ is a polynomial in k of degree $2h$ (lemma (3.3)).

Thus, the series $\sum_{i=n}^{\infty} \frac{1}{c_{i,h}}$ is convergent, which contradicts lemma (3.1). Therefore the inequality (3) has infinitely many solutions.

On the other hand, if the inequality (3) has infinitely many solutions then clearly the inequality (4) also has infinitely many solutions. The theorem is proved.

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References

[1] P. Erdos and P. Turan, On some new questions on the distribution of prime numbers , *Bull. Amer. Math. Soc.,* **54** (1948), 371 - 378.

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