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On Comparison of Spline Regularization and Fourier Series Methods for Inversion of Noisy Laplace Transforms

M. Iqbal

Department of Mathematics
University of Hail, Hail, Saudi Arabia
miqbal@uoh.edu.sa

Abstract

In this paper, we propose two numerical methods for computing a function given its Laplace transform function on the real axis. In the first method we have converted the Laplace transform to an integral equation of the first kind of convolution type, which is an ill-posed problem and used the spline regularization method to solve it. Inversion of noisy Laplace transform plays an important role in the system theory. In the second method the inversion algorithm is based on the Fourier series expansion of the unknown function and the Fourier coefficients are approximated using Tikhonov regularization. The methods are applied to several test examples taken from [3, 4, 15, 20, 30]. The results are shown in Tables 1 and Figs. (1-6).

Mathematics Subject Classification: 65R20, 65R30

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1. Introduction.

Noisy Laplace transforms arise in a wide variety of practical problems. It is frequently used in system theory and linear dynamical systems. It is also used in statistics where a sample is drawn from a cumulative distribution function G, which is an unknown mixture of exponential distributions and hence can be written as

$$G(t) = \int_0^\infty (1 - e^{-st}) f(s) ds, \quad t \in (0, \infty),$$

where f is a probability density function with support in $(0, \infty)$.

Chauveau, D.E. et al. [4] introduced the exponential sampling technique for inversion of the Laplace transform in photon correlation spectroscopy. There are many problems whose solution may be found in terms of a Laplace transforms which, however, is too complicated for inversion using different methods. However, no single method gives optimum results for all purposes and all occasions.

For a detailed bibliography, the reader should consult Piessens [24] and Piessens and Branders [25]. The problem of the recovery of a real function $\phi(t), \quad t \geq 0$, given its Laplace transform

$$\int_0^\infty e^{-st} \phi(t) dt = g(s) \tag{1.1}$$

for real values of s.

The Laplace transform inversion is an ill–posed problem and, therefore, affected by numerical instability. The ill–posedness of Laplace transform inversion in the case where $f \in L^2(R_+)$ and $g(s)$ is known for all real and positive values of s, can be investigated by means of the Mellin transform [20, 19]. In practice, however, $g(s)$ is known only in a finite set of points. The case of an infinite set of equidistant points was investigated by Papoulis [22]. Several methods and a comparison is given in [9, 28].

The previous methods do not include regularization techniques. Regularization methods have been discussed by Varah [30], Essah and Delves [13] and Chauveau [4] and Brianzi [3]. Regularization by means of truncated singular function expansion is investigated by Brianzi in [3]. Other methods are also available in the literature for the numerical evaluation of the Laplace transform inversion which have been described by Linz [19], Norden [21] and Salzer [26] and in [11, 14, 15, 12, 29].

2. Description of the First Method (Regularization Method)

In (1.1) given $g(s), \quad s \geq 0$ we wish to find $\phi(t), \quad t \geq 0$ and $\phi(t) = 0$ for $t < 0$, so that (1.1) holds. Frequently, $g(s)$ is measured at certain points. We assume $g(s)$ is given analytically with known $\phi(t)$, so that we can measure the error in the numerical solution. In order to convert the Laplace transform into the first kind integral equation of convolution type, we make the following substitution in equation (1.1).

$$s = a^x \text{ and } t = a^{-y} \text{ where } a > 1. \tag{2.1}$$

Then

$$g(a^x) = \int_{-\infty}^{\infty} (\log a) e^{-a^{x-y}} \phi(a^{-y}) a^{-y} dy. \tag{2.2}$$

Multiplying both sides of (2.2) by a^x we obtain the convolution equation

$$\int_{-\infty}^{\infty} K(x-y)F(y)dy = G(x), \quad -\infty \leq x \leq \infty \tag{2.3}$$

where

$$\left.\begin{array}{l} G(x) = a^x g(a^x) = sg(s) \\ K(x) = (\log a) a^x e^{-a^x} = (\log a) s e^{-s} \\ F(y) = \phi(a^{-y}) = \phi(t) \end{array}\right\} \tag{2.4}$$

In order that we can apply our deconvolution method to equation (2.3), it is necessary that $G(x)$ has essentially compact support, i.e. $G(x) \to 0$ as $x \to \pm\infty$ or $|\lambda - G(x)| \to 0$ as $x \to \pm\infty$ where $\lambda = \max |G(x)|$ as $x \to \pm\infty$ which is a property, we demand from our data function $G(x)$.

2.1 Tikhonov Regularization Using Cardinal Cubic B–splines

Let $B_j(H; x)$ be the N–th order cardinal B–spline (N even) with knotes $\left(j - \frac{N}{2}\right) H, \dots, \left(j + \frac{N}{2}\right) H$ i.e. $B_j(H; x) = Q_N\left(\frac{x}{H} - j + \frac{N}{2}\right)$ where

$$Q_N(x) = \frac{1}{(N-1)!} \sum_{j=0}^{N} (-1)^j \left(\begin{array}{c} N \\ j \end{array} \right) (x-j)_+^{N-1} \tag{2.5}$$

In addition let $MH = T$ where $M \leq N$ is an integral power of 2. We assume that $B_j(H; x)$ is periodically continued outside the interval $(0, T)$ with period T [27]. Then $B_j(H, x)$ has a Fourier series

$$B_j(H; x) = \sum_{q=-\infty}^{\infty} \hat{B}_{jq} \exp(i\omega_q x) \tag{2.6}$$

where

$$\hat{B}_{jq} = \left(\frac{1}{T}\right) \int_0^T B_j(H; x) \exp(-i\omega_q x) \tag{2.7}$$

and $\omega_q = \frac{2\pi q}{T}$.

Since $B_j(H;x)$ is simply a translation of $B_0(H;x)$ by an amount jH, we have

$$\hat{B}_{jq} = \hat{B}_{0,q} \exp(-i\omega_q H)$$

where

$$\hat{B}_{0,q} = H \left[\frac{\sin \frac{\omega_q H}{2}}{\frac{\omega_q H}{2}} \right]^4 \tag{2.8}$$

Now we shall approximate the convolution equation (2.3) by

$$\int_0^T K_N(x-y) F_M(y) dy = G_N(x) \tag{2.9}$$

where we assume that F, G and K have essentially finite support in $[0,T]$, F_M is a cubic spline $(N=4)$ of the form

$$F_M(x) = \sum_{j=0}^{M-1} \alpha_j B_j(H;x), \qquad M \leq N. \tag{2.10}$$

The real M–dimensional vector

$$\underline{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_{M-1})^T$$

of unknown coefficients needs to be determined.

The spline in equation (2.10) has the Fourier series

$$F_M(x) = \sum_{q=-\infty}^{\infty} \hat{F}_{M,q} \exp(i\omega_q x) \tag{2.11}$$

where

$$\left.\begin{aligned} \hat{F}_{M,q} &= \sum_{j=0}^{M-1} \alpha_j \hat{B}_{jq} \\ &= \hat{B}_{0,q} \sum_{j=0}^{M-1} \alpha_j \exp\left(-\frac{2\pi i}{M} jq\right) \\ &= \sqrt{M} \hat{B}_{0,q} \hat{\alpha}_s, \ s \approx q \ (\bmod M). \end{aligned}\right\} . \tag{2.12}$$

Here

$$\underline{\hat{\alpha}} = \psi_M^H \underline{\alpha} \tag{2.13}$$

where the symbol 'H' denotes the Hermitian transpose in (2.13).

We find it advantageous to determine $\underline{\hat{\alpha}}$ rather than $\underline{\alpha}$, because of the simple properties available in discrete Fourier spaces. The vector $\underline{\alpha}$ in equation (2.10) may then be determined from the inverse M–dimensional FFT (Fast Fourier Transform)

$$\underline{\alpha} = \psi_M \underline{\hat{\alpha}} \tag{2.14}$$

where ψ is the unitary matrix with elements

$$\psi_{rs} = \frac{1}{\sqrt{N}} \exp\left(\frac{2\pi i}{N} rs\right), \qquad r, s = 0, 1, 2, \ldots, N-1$$

2.2 P–th Order Tikhonov Regularization [29].

Consider the smoothing functional

$$C(F_M; \lambda) = C(\underline{\alpha}, \lambda) = \|K_N * F_M - G_N\|^2 + \lambda \|F_M^{(P)}\|^2 \tag{2.15}$$

where $F_M^{(P)}$ is the p–th order derivative.

Using Plancherel's theorem we have

$$\|K_N * F_M - G_N\|_2^2 = \frac{1}{N^2} \sum_{q=-N/2}^{N/2} |\hat{K}_{N,q} \hat{F}_{M,q} - \hat{G}_{N,q}|^2.$$

Hence using equation (2.12) we have

$$\|K_N * F_M - G_N\|_2^2$$

$$= \frac{1}{N^2} \sum_{q=-1/2N}^{1/2N} [(\sqrt{M} \hat{B}_{0,q} \hat{K}_{N,q} \hat{\alpha}_s - \hat{G}_{N,q})) \sqrt{M} \hat{B}_{0,q} \overline{\hat{K}}_{N,q} \overline{\hat{\alpha}}_s - \overline{\hat{G}}_{N,q})] \tag{2.16}$$

where $s \equiv q \pmod{M}$.

Also, Plancherel's theorem applied to the regularizing functional in equation (2.15) gives

$$\begin{aligned} \|F_M^{(P)}\|_2^2 &= \sum_{q=-\infty}^{\infty} \omega_q^{2p} |\hat{F}_{M,q}|^2 = 2 \sum_{q=1}^{\infty} \omega_q^{2p} |\hat{F}_{M,q}|^2 \\ &= 2M \sum_{q=1}^{\infty} \omega_q^{2p} \hat{B}_{0,q}^2 |\hat{\alpha}_s|^2 \qquad \text{where } s \equiv q \pmod{M} \end{aligned} \tag{2.17}$$

The simplification of expression (2.17) requires the use of an attenuation factor τ_q. For cubic cardinal splines it is shown by Stoer [27] and Gautschi [14] and Pennisi [23]. that

$$\tau_q = \left[\frac{\sin \frac{\pi q}{M}}{\frac{\pi q}{M}}\right]^4 \frac{3}{1 + 2\cos^2\left(\frac{\pi q}{M}\right)} \tag{2.18}$$

In expression (2.17) we wish to arrange the summation over q to summation over s where $s \equiv q(\bmod M)$.

Define the matrix

$$W^{(1)} = \begin{bmatrix} \text{diag} & \sqrt{M} & \hat{B}_{0,s} & \hat{K}_{N,s} \\ \cdots & \cdots & \cdots & \cdots \\ \text{diag} & \sqrt{M} & \hat{B}_{0,M-s} & \overline{\hat{K}}_{N,M-s} \end{bmatrix} \quad \begin{matrix} \text{order } N \times M \\ s = 0, 1, \ldots, M-1 \end{matrix} . \tag{2.19}$$

From the property $\hat{K}_{N,q} = \overline{\hat{K}}_{N,N-q}$ of discrete FTs, it then follows that expression (2.16) simplifies to

$$\|K * F_M - G_N\|^2 = \|W^{(1)}\underline{\hat{\alpha}} - \underline{\hat{G}}_N\|_2^2 \tag{2.20}$$

and (2.17) can be written as

$$\begin{aligned} \|F_M^{(p)}\|^2 &= 2M \sum_{s=1}^{M-1} \left\{ |\hat{\alpha}_s\|^2 \sum_{n=0}^{\infty} \omega_{Mn+s}^{2p} \hat{B}_{0,Mn+s}^2 \right\} \\ &= 2M \sum_{s=1}^{M-1} \tau_s |\hat{\alpha}_s|^2 \end{aligned} \tag{2.21}$$

where

$$\begin{aligned} \tau_s &= \sum_{n=0}^{\infty} \omega_{Mn+s}^{2p} \hat{B}_{0,Mn+s}^2 \qquad (2.22) \\ &= (2\pi)^{2p} \sum_{n=0}^{\infty} (Mn+s)^{2p} H^2 \left[\frac{\sin \pi \frac{(Mn+s)}{M}}{\frac{\pi(Mn+s)}{M}}\right] \\ \tau_s &= (2\pi)^{2p} H^2 s^8 \left[\frac{\sin \frac{\pi s}{M}}{\pi s/M}\right]^8 \sum_{n=0}^{\infty} \quad (Mn+s)^{2p-8} \\ &= (2\pi)^{2p} s^8 \hat{B}_{0,s}^2 \sum_{n=0}^{\infty} (Mn+s)^{2p-8} \qquad (2.23) \end{aligned}$$

since $\hat{\alpha}_s = \overline{\hat{\alpha}}_{M-s}$, equation (2.21) further simplifies to

$$\|F_M^{(p)}\|_2^2 = \sum_{s=1}^{1/2M} (\tau_s + \tau_{M-s}) |\hat{\alpha}_s|^2. \tag{2.24}$$

In particular, when $p = 2$ (the order of regularization from (2.23), it follows that

$$\tau_s = (2\pi)^4 s^4 \hat{B}_{0s}^2 \sum_{n=0}^{\infty} \left(\frac{s}{Mn + s} \right)^4$$

while

$$\tau_{M-s} = (2\pi)^4 s^4 \hat{B}_{0s}^2 \left(\frac{s}{Mn - s} \right)^4$$

so that

$$\begin{aligned}
\tau_s + \tau_{M-s} &= (2\pi)^4 s^4 \hat{B}_{0,s}^2 \sum_{n=-\infty}^{\infty} \left(\frac{s}{Mn + s} \right)^4 \\
&= \frac{(2\pi)^4 s^4 \hat{B}_{0,s}^2 \left[1 + 2\cos^2\left(\frac{\pi s}{M}\right) \right]}{3 \left[\frac{\sin \frac{\pi s}{M}}{\pi s / M} \right]^4} \qquad \text{see Pennisi [23])} \\
\tau_s + \tau_{M-s} &= \frac{16}{3} M^2 \sin^4\left(\frac{\pi s}{M} \right) \left[1 + 2\cos^2\left(\frac{\pi s}{M} \right) \right] \qquad (2.25)
\end{aligned}$$

Defining the $M \times M$ matrix

$$W^{(2)} = \text{diag}\,\{[M(\tau_s + \tau_{M-s})]^{1/2}\} \qquad (2.26)$$

it follows from (2.24) that

$$\|F_M^{(p)}\|_2^2 = \|W^{(2)} \underline{\hat{\alpha}}\|^2. \qquad (2.27)$$

Thus from equations (2.20) and (2.27) we may express the smoothing functional (2.15) as

$$C(\underline{\alpha}, \lambda) = \|W^{(1)} \underline{\hat{\alpha}} - \underline{\hat{G}}_N\|_2^2 + \lambda \|W^{(2)} \underline{\hat{\alpha}}\|_2^2. \qquad (2.28)$$

The minimizer of (2.28) is clearly

$$\underline{\hat{\alpha}} = (W + \lambda V)^{-1} W^{(1)H} \underline{\hat{G}}_N \qquad (2.29)$$

where

$$\left.\begin{aligned} W &= W^{(1)H} W^{(1)} \\ V &= W^{(2)H} W^{(2)} \end{aligned}\right\} \qquad (2.30)$$

It is not necessary to invert the matrix $W + \lambda V$ directly because it is diagonal. From equations (2.19), (2.26), (2.29) and (2.30), it follows that

$$\begin{aligned} \hat{\alpha}_s &= \frac{1}{\sqrt{M}} \frac{\hat{B}_{0,s}\overline{\hat{K}}_{N,s}\hat{G}_{N,s} + \hat{B}_{),M-s}\hat{K}_{N,M-s}\overline{\hat{G}}_{N,M=s}}{\left[\hat{B}^2_{0,s}\left\{|\hat{K}_{N,s}|^2 + \left(\frac{s}{M-s}\right)^8 |\hat{K}_{N,M-s}|^2\right\} + N^2\lambda(\tau_s + \tau_{M-s})\right]} \\ \hat{\alpha}_s &= \frac{1}{\sqrt{M}} \frac{\hat{B}_{),s}\left[\overline{\hat{K}}_{N,s}\hat{G}_{N,s} + \left(\frac{s}{M-s}\right)^4 \hat{K}_{N,M-s}\overline{\hat{G}}_{N,M-s}\right]}{\hat{B}^2_{0,s}\left[|\hat{K}_{N,s}|^2 + \left(\frac{s}{M-s}\right)^8 |\hat{K}_{N,M-s}|^2\right] + N^2\lambda(\tau_s + \tau_{M-s})} \end{aligned} \quad (2.31)$$

since

$$\hat{B}_{0,M-s} = \left(\frac{s}{Mn - s}\right)^4 \hat{B}_{0,s} \quad (2.32)$$

we can easily verify that $\hat{\alpha}_s = \overline{\hat{\alpha}}_{M-s}$ so that the inverse FFT gives $\underline{\alpha} = \psi_M \underline{\hat{\alpha}}$ is a real vector as required.

2.3 The Filter for Cardinal B–Spline Regularization.

The Fourier coefficients of the regularized (filtered) solution $F_M \in B_M(0,T)$ clearly depends on λ through equations (2.12), (2.13) and (2.31). In equation (2.31), we denote the dependence of $\hat{\alpha}_s$ on λ by writing $\hat{\alpha}_s = \hat{\alpha}_s(\lambda)$. Thus the Fourier coefficients of the filtered solution are

$$\hat{F}_{M,q}(\lambda) = \sqrt{M}\hat{B}_{0,q}\hat{\alpha}_s(\lambda), \quad s \equiv q(\bmod M)$$

whereas those of the unregularized (unfiltered) solution is

$$\hat{F}_{0,q}(0) = \sqrt{M}\hat{B}_{0,q}\hat{\alpha}_s(0)$$

Clearly the underlying filter $Z_{q;\lambda}$ must satisfy

$$\hat{F}_{M,q}(\lambda) = Z_{q;\lambda}\hat{F}_{M,q}(0)$$

so that we can deduce

$$Z_{q;\lambda} = \frac{\hat{\alpha}_s(\alpha)}{\hat{\alpha}_s(0)} \quad (2.33)$$

$$Z_{q;s} = \frac{\hat{B}^2_{0,s}\left[|\hat{K}_{N,s}|^2 + \left(\frac{s}{Mn-s}\right)^8 |\hat{K}_{N,M-s}|^2\right]}{\hat{B}^2_{0,s}\left[|\hat{K}_{N,s}|^2 + \left(\frac{s}{Mn-s}\right)^8 |\hat{K}_{N,M-s}|^2\right] + N^2\lambda(\tau_s + \tau_{M-s})} \quad (2.34)$$

The filter will of course apply to every Fourier coefficients $q = 0, \pm 1, \pm 2, \ldots$, but will have only M possible values depending upon q modulo M. The regularization parameter λ is still to be determined.

2.4 Determination of Regularization Parameter λ.

Let the filtered solution $F_M \in B_M(0,T)$, which minimizes $\|K_N * F_M - G_N\|_2^2 + \lambda \|F_M''\|_2^2$ be given by (we have $p = 2$)

$$F_M(x) = \sum_{q=-\infty}^{\infty} \hat{F}_{M,q} \exp(i\omega_q x) \tag{2.35}$$

Consider

$$\begin{aligned} \hat{G}_{N,\lambda,q} &= \hat{K}_{N,q}\hat{F}_{M,q}, \quad q = 0, 1, \ldots, N-1 \\ &= \begin{cases} \sqrt{M}\hat{B}_{0,1}\hat{K}_{N,q}\hat{\alpha}_s, & s \equiv q (\text{mod } M) \\ 0, & \text{otherwise} \end{cases} \quad \text{for } q = 0, 1, 2, \ldots N\text{–}1. \end{aligned} \tag{2.36}$$

We now introduce the $N \times N$ influence matrix

$$A(\lambda) = \psi_N \tilde{A}(\lambda) \psi_N^H$$

where

$$\underline{\hat{G}}_{N,\lambda} = \tilde{A}(\lambda) \underline{G}_N \tag{2.37}$$

$\hat{A}(\lambda)$ is block–diagonal with the following structure

$$\hat{A}(\lambda) \left[\begin{array}{c|c} \text{diag } \underline{a}_1 & \text{diag } \underline{a}_2 \\ \hline \text{diag } \underline{a}_3 & \text{diag } \underline{a}_4 \end{array} \right] \tag{2.38}$$

where $\underline{a}_k \in C^M$, $\quad K = 1, 2, 3, 4$ and

$$a_{1,s} = \begin{cases} \dfrac{\sqrt{M}(\hat{B}_{0,s})^2 |\hat{K}_s|^2}{D_s}, & s = 0 \\ \dfrac{\sqrt{M}(\hat{B}_{0,s})^2 |\hat{K}_s|^2}{2D_s}, & 1 \le s \le M-1 \end{cases}$$

$$a_{2,s} = \begin{cases} 0, & s = 0 \\ \dfrac{\sqrt{M}(\hat{B}_{0,s})^2 \left(\frac{s}{M-s}\right)^4}{\hat{K}} \, \overline{\hat{K}}_{M+s}, & 1 \le s \le M-1 \\ \quad s & \\ \dfrac{\sqrt{M}\hat{K}_{M+s}\hat{B}_{0,M+s}\hat{B}_{0,s}\overline{\hat{K}}_s}{D_s}, & s = 0 \\ \dfrac{\sqrt{M}\hat{K}_{M+s}\hat{B}_{0,M+s}\hat{B}_{0,s}\overline{\hat{K}}_s}{2D_s}, & 1 \le s \le M-1 \end{cases}$$

$$a_{4,s} = \begin{cases} 0, & s = 0 \\ \dfrac{\sqrt{M}\hat{B}_{0,M+s}\hat{B}_{0,s}\left(\frac{s}{M+s}\right)^4 |\hat{K}_{M+s}|^2}{2D_s}, & 1 \le s \le M-1 \end{cases}$$

where

$$D_s = M\hat{B}_{0,s}^2 \left[|\hat{K}_s|^2 + \left(\frac{s}{M-s} \right)^8 |\hat{K}_{M-s}|^2 \right] + \lambda N^2(\tau_s + \tau_{M+s}).$$

For simplicity of notation we have written $\hat{K}_s$ for $\hat{K}_{N,s}$ in $a_{1,s}, a_{2,s}, a_{3,s}, a_{4,s}$ and D_s. The optimal λ as defined by GCV method may be found in Wahba [31]. Now minimizing the expression

$$V(\lambda) = \frac{\frac{1}{N} \| I - \hat{A}(\lambda) \underline{\hat{G}}_N \|_2^2}{\left[\frac{1}{N} \text{ Trace}(I - \hat{A}(\lambda)) \right]^2} \tag{2.39}$$

which from equation (2.38) can be written as

$$V(\lambda) = \frac{\frac{1}{N} \left\{ \sum_{s=0}^{M-1} |(1 - a_{1,s})\hat{G}_s - a_{2,s}\overline{\hat{G}}_{M-s}|^2 + \sum_{s=0}^{M-1} |(1 - a_{4,s})\overline{\hat{G}}_{M-s} - a_{3,s}\hat{G}_s|^2 \right\}}{\left[1 - \frac{1}{N} \sum_{s=0}^{M-1} (a_{1,s} + a_{4,s}) \right]^2}. \tag{2.40}$$

In order to minimize $V(\lambda)$ in equation (2.40) we have used a subroutine which uses a quadratic interpolation technique to obtain a minimum.

3. Addition of Random Noise to the Data Functions.

In solving the problems (1–5), we have considered the data functions contaminated by varying amounts of random noise. To generate sequences of random errors of the form $\epsilon_n, n = 0, 1, 2, \ldots, N-1$, we have used a subroutine which returns pseudo–random real numbers taken from a normal distribution of prescribed mean A and standard deviation B.

To mimic experimental errors we have

$$A = 0 \text{ and } B = \left(\frac{X}{100} \right) \max_{0 \leq n \leq N-1} |G_n| \tag{3.1}$$

where X denotes a chosen percentage. In all our test problems we have taken $x = 0.7$ because Laplace transform is a severally ill–posed problem. Thus the random error ϵ_n added to G_n does snot exceed $3X\%$ of the maximum value of $G(x)$.

4. Optimal Convergence.

Here we shall discuss the optimal convergence to the case of convolution equation (2.9).

$$\int_0^T K(x-y)F(y)dy = G(x),$$

in which the function K is periodic of period T, expanding K in a Fourier series

$$K(x-y) = \sum_{q=-\infty}^{\infty} \hat{K}_q \exp\left(\frac{2\pi i}{T} q(x-y)\right), \quad \text{In [2, 5, 6, 8, 10, 11, 12]} \tag{4.1}$$

where $\hat{K}_q$ is the Fourier coefficient, i.e.

$$\hat{K}_q = \int_0^T K(y) \exp\left(\frac{2\pi i}{T} qy\right) dy = \overline{\hat{K}}_{N-q} \tag{4.2}$$

We shall assume that

$$|\hat{K}_q| \simeq q^{-K}, \quad K > 1$$

and so the Fourier series is uniformly convergent.

5. Method 2 (Fourier Series Method [6])

The Laplace transform $g(s)$ in terms of the basis functions $p_k(s)$ depends on the discretization step size, $h = \dfrac{\pi}{T}$ is used to minimize the discretization error $\delta(s)$. When $p_k(s)$ have been selected, then the first $m+1$ coefficients in $a_k\,, \quad k = 0, 1, 2, \ldots, m$

$$g(s) = \sum_{k=0}^{\infty} a_k p_k(s) + \delta(s), \quad k = 0, 1, \ldots, m \tag{A}$$

can be computed as

$$a_k, k = 0, 1, \ldots, m = \underset{c_k \in R^{m+1}}{\arg\min} \left\| g(s) - \sum_{k=1}^{\infty} c_k p_k(s) \right\|_{L^2(0,T)} \tag{5.1}$$

where m is the discretization parameter and $\|\cdot\|_{L^2(0,T)}$ is the L^2 norm, where the coefficient a_0 is

$$a_0 = c_0 = \left(\frac{1}{T}\right) \cup (d, 0)$$

as the first term of the sum is known and the minimization problem can be rewritten as follows:

$$a_k, k = 0, 1, 2, \ldots, m = \underset{c_k \in R^m}{\arg\min} \left\| g(s) - \sum_{k=1}^{m} c_k p_k(s) \right\|_{L^2(0,T)} \tag{5.2}$$

Actually due to the intrinsic ill-posedness of the problem, a straightforward approach is not recommended, a reasonable way to compute the stable solution is to regularize the problem. More precisely by applying Tikhonov regularization [29, 7, 8], we compute the approximations

$a_k^\lambda, k = 1, 2, \ldots, m$ of the m coefficients $a_k, k = 1, 2, \ldots, m$ in (A), as (B)

$$a_k^\lambda, k = 1, 2, \ldots, m = \underset{c_k \in R^m}{\arg\min} \left\| g(s) - \sum_{k=1}^{m} c_k p_k(s) \right\|_2^2 + \lambda \|C\|_{L^2(0,T)} \tag{5.3}$$

where $\lambda > 0$ is the regularization parameter.

The common approach to numerically solving (B) is first to discretize continuous problem and then to regularize the finite dimensional problem.

Let $s = (s_1, s_2, \ldots, s_n)$ be n values on the real axis, and $G = (g(s_1), g(s_2), \ldots, g(s_n) - (a_0 p_0(s_1) + a_0 p_0(s_2) + \cdots + a_0 p_0(s_n)))$ be the n dimensional vector whose i-th component is the value of the function $(g(s) - a_0 p_0(s)$ at s.

Let $B = (b_{ki}), k = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, n$, be the matrix arising by collecting the basic function $p_k(s)$ at s, i.e., $p_k(s) = b_{ki}, k = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, n$. The discretization gives rise to the following linear system

$$BC = G, \quad G \in R^n \ \text{ and } \ B \in R^{m \times n}, \quad C \in R^n. \tag{5.4}$$

Tikhonov regularization to (5.4) can be regarded as the following squares problem

$$\min_{c \in R^m} \left\| \begin{pmatrix} B \\ \sqrt{\lambda}\, I \end{pmatrix} C - \begin{pmatrix} G \\ 0 \end{pmatrix} \right\|_2^2 \leftrightarrow \min_{c \in R^m} \left[\|BC - G\|_2^2 + \lambda \|C\|_2^2 \right]$$

where $\|\cdot\|$ is the euclidean vector norm, [see 10, 11].

One of the most important problems related to the numerical treatment of regularization methods is the choice of the regularization parameter λ. One of the most used methods is the generalized crossvalidation (GCV) method [31].

Error Analysis

Now in the following lemma, we dicuss in it, which show the connection between the regularization parameter and the conditioning of the regularized problem (see [7, 8, 16, 17, 18]).

Lemma. *Let f_λ denot the solution of*

$$\min \|Kf - g\|_2^2 + \lambda^2 \|f\|_2^2$$

and let $\overline{f}_\lambda$ denote the solution to

$$\min \|Kf - \overline{g}\|_2^2 + \lambda^2 \|f\|_2^2.$$

Then

$$\frac{\|f^\lambda - \overline{f}^\lambda\|_2^2}{\|f^\lambda\|_2^2} \leq \frac{\sigma_1 \|e\|_2^2}{2\lambda \|b^\lambda\|_2^2}$$

where $b^\lambda = Kf^\lambda$ and $e = g - \overline{g}$.

Numerical Examples

In this section we calculate results of the above methods to test problems over the interval $[0, T]$. All data functions have property $g(s) = 0(s^{-1})$ and so we deal with severely ill-posed problems. To make the problems more ill-posed apart from machine rounding error, 0.7% noise is added. In method 1 in all cases we have taken $N = 64$ data points.

Test Problems

Problem 1. The problem has been taken from Varah [30].

$$\begin{aligned} g(s) &= \frac{1/2}{s(s+1/2)} \\ f(t) &= 1 - e^{-t/2}. \end{aligned}$$

The optimal results for method 1 are shown in Table 1.

Problem 2. The problem has been taken from McWhirter [20] and Brianzi [3]

$$\begin{aligned} g(s) &= \frac{1}{(s+1)^2} \\ f(t) &= te^{-t}. \end{aligned}$$

The optimal results for method 1 are shown in Table 1.

Problem 3. This problem has been taken from Cristina [4] and Giunta [15]

$$
\begin{aligned}
g(s) &= \frac{\beta}{(s+\alpha)^2+\beta^2} \\
f(t) &= e^{-\alpha t}\sin(\beta t) \quad \text{for} \quad \alpha = 1/2 \quad \text{and} \quad \beta = \frac{\sqrt{3}}{2}.
\end{aligned}
$$

The optimal results for method 1 are shown in Table 1.

Table 1

Test Problem	α	T	h	λ	$V(\lambda)$	$\|f-f_\lambda\|_\infty$	Fig.
1	10.0	9.60	0.150	0.31×10^{-8}	0.56358	0.01	1
2	10.0	11.60	0.18125	0.11×10^{-8}	0.1167×10^{-8}	0.010	2
3	8.0	12.60	0.19688	0.14×10^{-8}	0.1013×10^{-9}	0.007	3

Numerical Results

In this section we tabulate the results of Method 1, applied to the test problems taken from the literature [3, 4, 15, 20, 30]. All data functions have property $g(s) = 0(s^{-1})$ and 0.7% noise has been added, apart from the machine rounding error.

Only optimal results have been quoted in Table 1, and demonstrated in the respective figures. In each of the test problems 64 sample points are used to calculate discrete Fourier coefficients.

In Method 1, for numerical calculations, we need to choose two numbers $x_{\max}$ and $x_{\min}$, we find $x_{\max}$ and $x_{\min}$ as the largest and smallest solutions of the nonlinear equation $G(x) = \epsilon$ where $\epsilon = 10^{-4}$. We may then pose the deconvolution problem (2.3) on the interval $[0, T]$ where $T = x_{\max} - x_{\min}$. Since the size of the essential support of $G(x)$ depends upon 'α' we have for a fixed number N of equidistant data points $\{x_n\}$, where $h = T/N$. We have minimized (2.40) with respect to λ for values of $\alpha > 1$ and compared the L_∞ error of the resulting solution with the values of the true solution. To calculate the optimal value of regularization parameter λ, we minimize value of $V(\lambda)$ that yields the L_∞ error of the regularized solution.

Conclusion

Method 1 and Method 2 worked very well over the three test problems. The results obtained are shown in Table 1. Table of values for Method 2 is not available to compare with, but could be computed easily.

Method 1 has an edge over Method 2 because using Method 1 we added noise about 0.7% to make the problems more ill-posed whereas Method 2 has been applied by the author [7, 8], on clean data.

Also we can compare with the results of the problems from the literature, which show that Method 1 worked very well over the test problems borrowed from Varah [30], McWhirter and Pike [20] and Brianzi [3].

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