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Comparability of Some Classical Integrals in Weighted Lebesgue Spaces

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To the Memory of Professor Todor Genchev

Abstract

Modified versions of the Riemann-Liouville- and Weyl fractional integral operators are shown to be comparable in $L^p_w(0,\infty)$, $1 < p < \infty$, where w belongs A_p-weight classes. In addition, we utilize the boundedness of these operators on $L^p_w(\mathbb{R}^n)$ to derive weight characterizations for which higher order differential inequalities are satisfied in weighted Lebesgue spaces. These results extend the weight Sobolev type gradient inequalities proved in [5] via the Hardy operator.

Keywords: Weighted norm inequalities, A_p-weights, Comparability, Riemann-Liouville-, Weyl fractional integrals, Differential inequalities, Weighted Lebesgue spaces

1 Introduction

Let I_α, J_α, $\alpha \geq 0$, denote the modified Riemann-Liouville- and Weyl fractional integral operators defined by

$$(I_\alpha f)(x) = \frac{\alpha+1}{x^{\alpha+1}} \int_0^x (x-y)^\alpha f(y)dy \ ;$$
$$(J_\alpha f)(x) = (\alpha+1) \int_x^\infty (y-x)^\alpha y^{-\alpha-1} f(y)dy \ , \tag{1.1}$$

where $x > 0$.

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If $\alpha = 0$, these operators reduce to the Hardy averaging operator and its conjugate. In this case $I_0(J_0 f) = I_0 f + J_0 f$, from which it follows (cf. [2, Theorem 2]) that for $f \geq 0$, I_0 and J_0 are comparable in weighted Lebesgue spaces with weights in the S_p-class. That is, if $w \in A_p$, $1 < p < \infty$, then there are constants $C_i > 0$, $i = 0, 1$, such that,

$$C_0 \leq \|I_0 f\|_{p,w} / \|J_0 f\|_{p,w} \leq C_1 \ .$$

This relationship is also denoted by $\|I_0 f\|_{p,w} \approx \|J_0 f\|_{p,w}$, and similarly for I_α, J_α and other entities.

Together with some preliminary results we prove in the next section that I_α and J_α, $\alpha \in \mathbb{N}$ are comparable in weighted Lebesgue spaces. In Section 3 we see that a change of variable in the definition of I_α and J_α makes it possible to define the operators of (1.1) also for functions f on $\mathbb{R}^n$. It is then possible to show that these extended operators (denoted by $\bar{I}_\alpha, \bar{J}_\alpha$) are bounded from $L^p_v(\mathbb{R}^n)$ to $L^p_u(\mathbb{R}^n)$, if and only if, the weight functions u and v satisfy certain conditions. In the case $u = v$ they include the familiar A_p-weight classes. Furthermore, we derive comparability results for these extended operators. In the final section we prove from the weight characterizations of $\bar{I}_1$ and $\bar{J}_1$ on $L^p_w(\mathbb{R}^n)$ differential inequalities in weighted Lebesgue spaces.

As usual, inequalities such as (2.1) are interpreted in the sense that if the right side is finite, so is the left side and the inequality holds. Functions are assumed to be measurable and weight functions are positive and locally integrable. $f \in L^p_v(\mathbb{R}^n)$ means that $f v^{1/p} \in L^p(\mathbb{R}^n)$ with norm $\|f\|_{p,v} = \|v^{1/p} f\|_p$. Expressions of the form $0 \cdot \infty$ are taken to be zero, while constants are denoted by C, sometimes with sub- and superscripts, which may be different at different occurrences. The conjugate index of $p > 1$ is defined by $p' = p/(p-1)$ and similarly for q. $C^k_0(\mathbb{R}^n)$, $k = 1, 2, \ldots$, are the classes of k times continuously differentiable functions of compact support. Other notation and conventions are standard and are introduced as they arise.

2 Comparability of I_α and J_α, $\alpha \in \mathbb{N}$

In order to prove comparability, we require some known weight characterizations for which the operators I_α, J_α, $\alpha \geq 0$, are bounded in weight Lebesgue spaces. These results were proved in 1989 by F.J. Martin-Reyes and E. Sawyer [3] and independently by V.D. Stepanov [6,7] with different (but equivalent) weight conditions. We state here the result given by Stepanov as they apply to I_α and J_α.

Theorem 2.1 *([6,7]) Suppose* $1 < p \leq q < \infty$ *and* u, v *weight functions on* $(0, \infty)$.

(a) The operator I_α, $\alpha \geq 0$, satisfies

$$\|I_\alpha f\|_{q,u} \leq C_0 \|f\|_{p,v} \ , \tag{2.1}$$

if and only if $D = \max(D_1, D_2) < \infty$, where

$$D_1 = \sup_{t>0} \left(\int_t^\infty (x-t)^{\alpha q} x^{-(\alpha+1)q} u(x) dx \right)^{1/q} \left(\int_0^t v(x)^{1-p'} dx \right)^{1/p'}$$

and

$$D_2 = \sup_{t>0} \left(\int_t^\infty x^{-(\alpha+1)q} u(x) dx \right)^{1/q} \left(\int_0^t (t-x)^{\alpha p'} v(x)^{1-p'} dx \right)^{1/p'} .$$

(b) The operator J_α, $\alpha \geq 0$, satisfies

$$\|J_\alpha f\|_{q,u} \leq C_1 \|f\|_{p,v} \tag{2.2}$$

if and only if $D^* = \max(D_1^*, D_2^*) < \infty$, where

$$D_1^* = \sup_{t>0} \left(\int_0^t (t-x)^{\alpha q} u(x) dx \right)^{1/q} \left(\int_t^\infty x^{-p'(\alpha+1)} v(x)^{1-p'} dx \right)^{1/p'}$$

and

$$D_2^* = \sup_{t>0} \left(\int_0^t u(x) dx \right)^{1/q} \left(\int_t^\infty (x-t)^{\alpha p'} v(x)^{1-p'} dx \right)^{1/p'} .$$

Moreover, $C_0 \approx D$ and $C_1 \approx D^*$.

Note that in the case $\alpha = 0$, $D_1 = D_2$ and $D_1^* = D_2^*$. Hence in the case of the Hardy averaging operator Theorem 2.1(a) yields:

$$\|I_0 f\|_{q,u} \leq C_0 \|f\|_{p,v} \ , \tag{2.3}$$

if and only if,

$$\sup_{t>0} \left(\int_0^\infty \frac{u(x)}{x^q} \, dx \right)^{1/q} \left(\int_0^t v(x)^{1-p'} dx \right)^{1/p'} < \infty \ . \tag{2.4}$$

Similarly, in the case of the conjugate Hardy operator, Theorem 2.1(b) yields

$$\|J_0 f\|_{q,u} \leq C_1 \|f\|_{p,v} \ , \tag{2.5}$$

if and only if,

$$\sup_{t>0} \left(\int_0^t u(x) dx \right)^{1/q} \left(\int_t^\infty \frac{v(x)}{x^{p'}} dx \right) < \infty \ . \tag{2.6}$$

We now recall the definition of the weight classes A_p and B_p.

Definition 2.1 *(a) A weight function w defined on $(0,\infty)$ belongs to A_p, $1 < p < \infty$, if for all $t > 0$, there is a constant $C > 0$, such that,*

$$\left(\int_0^t w(x)dx\right)^{1/p}\left(\int_0^t w(x)^{1-p'}dx\right)^{1/p'} \le Ct \tag{2.7}$$

is satisfied.

(b) $w \in B_p$, $1 < p < \infty$, if for all $t > 0$, there is a constant $C > 0$, such that,

$$\int_t^\infty \frac{w(x)}{x^p}\,dx \le Ct^{-p}\int_0^t w(x)dx \tag{2.8}$$

holds.

Remark 2.1 *(a) It is well known (cf. [1]) that $A_p \subset B_p$, $1 < p < \infty$. Moreover, a simple observation shows that $w \in A_p$, if and only if, $w^{1-p'} \in A_{p'}$. It follows therefore that in case $u = v \equiv w$ and $q = p$, condition (2.4) holds if $w \in Z_p$ and similarly condition (2.6) is satisfied if $w \in A_p$. Hence, if $w \in A_p$, both the Hardy averaging operator and its conjugate are bounded on L^p_w.*

(b) This observation extends to the operators I_α and J_α, $\alpha \ge 0$. Clearly D_1 and D_2, respectively, D_1^* and D_2^* of Theorem 2.1 are dominated by the supremum of (2.4), respectively, the supremum of (2.6). Hence we obtain from Theorem 2.1 in the case $q = p$ and $u = v \equiv w$ the following.

Corollary 2.1 *If $w \in A_p$, $1 < p < \infty$, then for $\alpha \ge 0$*

$$\|I_\alpha f\|_{p,w} \le C_0\|f\|_{p,w} \quad \text{and} \quad \|J_\alpha f\|_{p,w} \le C_1\|f\|_{p,w}\ . \tag{2.9}$$

The main result of this section is now

Theorem 2.2 *If $f \ge 0$ and $w \in A_p$, $1 < p < \infty$, then*

$$\|I_\alpha f\|_{p,w} \approx \|J_\alpha f\|_{p,w} \tag{2.10}$$

for $\alpha \in \mathbb{N}$.

Proof. It suffices to prove the result for compactly supported f, for then standard limiting arguments yield the result.

If $\alpha \in \mathbb{N}$, then repeated integration by part shows that

$$\begin{aligned}(I_\alpha f)(x) &= \frac{\alpha+1}{x^{\alpha+1}}\int_0^x (x-y)^\alpha f(y)dy \\ &= \frac{\alpha+1}{x^{\alpha+1}}\int_0^x (x-y)^\alpha y^{\alpha+1} d\left(-\int_y^\infty t^{-\alpha-1}f(t)dt\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha+1}{x^{\alpha+1}}\left\{-(x-y)^{\alpha}y^{\alpha+1}\int_y^{\infty} t^{-\alpha-1}f(t)dt\Big|_0^x\right. \\
&\quad \left. + \int_0^x \frac{d}{dy}[(x-y)^{\alpha}y^{\alpha+1}]\left(\int_y^{\infty} t^{-\alpha-1}f(t)dt\right)dy\right\} \\
&= \frac{\alpha+1}{x^{\alpha+1}}\int_0^x \frac{d}{dy}[(x-y)^{\alpha}y^{\alpha+1}]d\left(-\int_y^{\infty}\int_t^{\infty} s^{-\alpha-1}f(x)dsdt\right) \\
&= \frac{\alpha+1}{x^{\alpha+1}}\left\{-\frac{d}{dy}[(x-y)^{\alpha}y^{\alpha+1}]\int_y^{\infty} s^{-\alpha-1}(s-y)f(s)ds\Big|_0^x\right. \\
&\quad \left. + \int_0^x \frac{d^2}{dy^2}[(x-y)^{\alpha}y^{\alpha+1}]\left(\int_y^{\infty} s^{-\alpha-1}(s-y)f(s)ds\right)dy\right\} \\
&= \cdots = \frac{\alpha+1}{x^{\alpha+1}}\left\{-\frac{d^{\alpha}}{dy^{\alpha}}[(x-y)^{\alpha}y^{\alpha+1}]\int_y^{\infty} s^{-\alpha-1}\frac{(s-y)^{\alpha}}{\alpha!}f(s)ds\Big|_0^x\right. \\
&\quad \left. + \int_0^x \frac{d^{\alpha+1}}{dy^{\alpha+1}}[(x-y)^{\alpha}y^{\alpha+1}]\left(\int_y^{\infty} \frac{(s-y)^{\alpha}}{\alpha!}s^{-\alpha-1}f(s)ds\right)dy\right\}. \qquad (2.11)
\end{aligned}
$$

But since

$$
\frac{d^{\alpha}}{dy^{\alpha}}[(x-y)^{\alpha}y^{\alpha}] = \sum_{k=0}^{\alpha} C_{\alpha-k}y^{\alpha+1-k}(x-y)^k
$$

for some constants $C_{\alpha-k}$, $k = 0, 1, \ldots, \alpha$; the integrated term in (2.11) will be zero for all except the $k = 0$ term in this sum. Also

$$
\frac{d^{\alpha+1}}{dy^{\alpha+1}}[(x-y)^{\alpha}y^{\alpha+1}] = \sum_{k=0}^{\alpha} C'_k y^k (x-y)^{\alpha-k}
$$

for some constants C'_k, $k = 0, 1, \ldots, \alpha$; Hence

$$
\begin{aligned}
(I_{\alpha}f)(x) &= \frac{\alpha+1}{x^{\alpha+1}}\left\{\frac{-C_{\alpha}x^{\alpha+1}}{\alpha!}\int_x^{\infty} s^{-\alpha-1}(s-x)^{\alpha}f(s)ds\right. \\
&\quad \left. + \sum_{k=0}^{\alpha}\frac{C'_k}{\alpha!}\int_0^x y^k(x-y)^{\alpha-k}\left(\int_y^{\infty}(s-y)^{\alpha}s^{-\alpha-1}f(s)ds\right)dy\right\} \\
&= \frac{-C_{\alpha}}{\alpha!}(J_{\alpha}f)(x) + \frac{1}{\alpha! x^{\alpha+1}}\sum_{k=0}^{\alpha} C'_k \int_0^x y^k (x-y)^{\alpha-k}(J_{\alpha}f)(y)dy
\end{aligned}
$$

and therefore,

$$
\begin{aligned}
|(I_{\alpha}f)(x)| &\le \frac{|C_{\alpha}|}{\alpha!}|(J_{\alpha}f)(x)| + \frac{1}{\alpha! x}\sum_{k=0}^{\alpha}|C'_k|\int_0^x |(J_{\alpha}f)(y)|dy \\
&\le C|(J_{\alpha}f)(x)| + C'(I_0|(J_{\alpha}f)|)(x)\ .
\end{aligned}
$$

Now by Minkowski's inequality and Corollary 2.1 with $\alpha = 0$

$$\begin{aligned} \|I_\alpha f\|_{p,w} &\leq C\|J_\alpha f\|_{p,w} + C'\|I_0(|J_\alpha f|)\|_{p,w} \\ &\leq C_0\|J_\alpha f\|_{p,w} \ , \end{aligned}$$

which proves the first part of the theorem.

To prove that $\|J_\alpha f\|_{p,w} \leq C_1\|f\|_{p,w}$, $w \in A_p$ we proceed essentially as above. We give the argument here for completeness only. Again, repeated integration by parts yields

$$\begin{aligned} (J_\alpha f)(x) &= (\alpha+1)\int_x^\infty (y-x)^\alpha y^{-\alpha-1} f(y)dy \\ &= (\alpha+1)\int_x^\infty (y-x)^\alpha y^{-\alpha-1} d\left(\int_0^y f(t)dt\right) \\ &= (\alpha+1)\left\{(y-x)^\alpha y^{-\alpha-1}\left(\int_0^y f(t)dt\right)\Big|_x^\alpha \right. \\ &\quad \left. - \int_x^\infty \frac{d}{dy}[(y-x)^\alpha y^{-\alpha-1}]\left(\int_0^y f(t)dt\right)dy\right\} \\ &= (-1)(\alpha+1)\int_x^\infty \frac{d}{dy}[(x-y)^\alpha y^{-\alpha-1}] d\left(\int_0^y (y-t)f(t)dt\right) \\ &= \cdots = (-1)^\alpha(\alpha+1)\left\{\frac{d^\alpha}{dy^\alpha}[(x-y)^\alpha y^{-\alpha-1}]\left(\int_0^y \frac{(y-t)^\alpha}{\alpha!}f(t)dt\right)\Big|_x^\infty \right. \\ &\quad \left. - \int_x^\infty \frac{d^{\alpha+1}}{dy^{\alpha+1}}[(x-y)^\alpha y^{-\alpha-1}]\left(\int_0^y \frac{(y-t)^\alpha}{\alpha!}f(t)dt\right)dy\right\} . \end{aligned}$$

But since

$$\frac{d^\alpha}{dy^\alpha}[(x-y)^\alpha y^{-\alpha-1}] = \sum_{k=0}^\infty C_k y^{-\alpha-1-k}(x-y)^k$$

and

$$\frac{d^{\alpha+1}}{dy^{\alpha+1}}[(x-y)^\alpha y^{-\alpha-1}] = \sum_{k=0}^\alpha C'_k y^{-\alpha-2-k}(x-y)^k$$

for some constants C_k, C'_k, $k = 0, 1, \ldots, \alpha$; it follows that

$$\begin{aligned} (J_\alpha f)(x) &= (-1)^{\alpha+1}\frac{(\alpha+1)}{\alpha!}\left\{C_0 x^{-\alpha-1}\int_0^x (x-t)^\alpha f(t)dt \right. \\ &\quad \left. + \sum_{k=0}^\alpha C'_k \int_x^\infty y^{-\alpha-2-k}(x-y)^k\left(\int_0^y (y-t)^\alpha f(t)dt\right)dt\right\} \\ &= (-1)^{\alpha+1}\frac{C_0}{\alpha!}(I_\alpha f)(x) + \sum_{k=0}^\alpha C'_k \int_x^\infty y^{-1-k}(x-y)^k (I_\alpha f)(y)dy \ . \end{aligned}$$

Therefore $|(I_\alpha f)(x)| \le C|(I_\alpha f)(x)| + C' J_0(|(I_\alpha f|)(x)$ and by Minkowski's inequality together with Corollary 2.1, with $\alpha = 0$, yields

$$\begin{aligned} \|J_\alpha f\|_{p,w} &\le C\|I_\alpha f\|_{p,w} + C'\|J_0(|I_\alpha f|)\|_{p,w} \\ &\le C_1\|I_\alpha f\|_{p,w} \; . \end{aligned}$$

This proves the theorem.

We have seen in Remark 2.1 that A_p implies that both D and D^* are finite in the case $q = p$ and $u = v \equiv w$. In the case $\alpha = 1$ we can replace the A_p weight of Theorem 2.2 by somewhat weaker conditions.

Corollary 2.2 *Suppose $f \ge 0$, $1 < p < \infty$ and D_1, D_2, D_1^*, D_2^* are the terms of Theorem 2.1 with $p = q$ and $u = v \equiv w$.*

(a) If $D = \max(D_1, D_2) < \infty$, then $\|I_1 f\|_{p,w} \le C_0\|J_1 f\|_{p,w}$.
(b) If $D^* = \max(D_1^*, D_2^*) < \infty$, then $\|J_1 f\|_{p,w} \le C_1\|I_1 f\|_{p,w}$.

Moreover, if D and D^* are finite, then $\|I_1 f\|_{p,w} \approx \|J_1 f\|_{p,w}$.

Proof. (a) Integration by parts and the fact that $f \ge 0$ shows that

$$\begin{aligned} (I_1 f)(x) &= 3I_1(J_1 f)(x) - 4I_0(J_1 f)(x) + (J_1 f)(x) \\ &\le 3I_1(J_1 f)(x) + (J_1 f)(x) \; . \end{aligned}$$

Minkowski's inequality and Theorem 2.1(a) with $\alpha = 1$, $u = v \equiv w$ proves (a).

To prove (b) one obtains similarly

$$\begin{aligned} (J_1 f)(x) &= (I_1 f)(x) + 3J_1(I_1 f)(x) - 4J_0(I_1 f)(x) \\ &\le (I_1 f)(x) + 3J_1(If)(x) \; . \end{aligned}$$

Minkowski's inequality and Theorem 2.1(b) then yields the result.

3 Generalization to $\mathbb{R}^n$

If one makes the substitution $y = xt$ in the definition of $I_\alpha f$ and $J_\alpha f$ in (1.1) then the operators take the form

$$(I_\alpha f)(x) = (\alpha{+}1)\int_0^1 (1{-}t)^\alpha f(xt)dt\,; \quad (J_\alpha f)(x) = (\alpha{+}1)\int_1^\infty (t{-}1)^\alpha t^{-\alpha-1} f(xt)dt\;.$$

Now if $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ these expressions are still meaningful. This leads to the following

Definition 3.1 *If f is defined on $\mathbb{R}^n$, then the extended operators $\bar{I}_\alpha, \bar{J}_\alpha$, $\alpha \geq 0$ are defined by*

$$(\bar{I}_\alpha f)(x) = (\alpha{+}1)\int_0^1 (1{-}t)^\alpha f(xt)dt\ ; \quad (\bar{J}_\alpha f)(x) = (\alpha{+}1)\int_1^\infty (t{-}1)^\alpha t^{-\alpha-1} f(tx)dt\ ,$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

The weight characterizations of Theorem 2.1 extend to this general framework in the case $q = p$.

Theorem 3.1 *Let $1 < p < \infty$ and u, v weight functions defined on $\mathbb{R}^n$. If $x \in \mathbb{R}^n$, and $x > 0$, write $u_{n,x}(s) = s^{n-1}u(sx)$ and $v_{n,x}(s) = s^{n-1}v(sx)$.*

(a) The operator $\bar{I}_\alpha$, $\alpha \geq 0$, satisfies

$$\|\bar{I}_\alpha f\|_{p,u} \leq C_0 \|f\|_{p,v}\ , \tag{3.1}$$

if and only if, $\bar{D} = \max(\bar{D}_1, \bar{D}_2) < \infty$, where

$$\bar{D}_1 = \sup_{x\in\mathbb{R}^n} \left(\int_1^\infty (s-1)^{\alpha p} s^{-(\alpha+1)p} u_{n,x}(s)ds\right)^{1/p} \left(\int_0^1 [v_{n,x}(s)]^{1-p'} ds\right)^{1/p'}$$

and

$$\bar{D}_2 = \sup_{x\in\mathbb{R}^n} \left(\int_1^\infty s^{-(\alpha+1)} u_{n,x}(s)ds\right)^{1/p'} \left(\int_0^1 (1-s)^{\alpha p'} v_{n,x}(s)^{1-p'} ds\right)^{1/p'} .$$

(b) The operator $\bar{J}_\alpha$, $\alpha \geq 0$, satisfies

$$\|\bar{J}_\alpha f\|_{p,u} \leq C_1 \|f\|_{p,v}\ , \tag{3.2}$$

if and only if $\bar{D}^* = \max(\bar{D}_1^*, \bar{D}_2^*), \infty$, where

$$\bar{D}_1^* = \sup_{x\in\mathbb{R}^n} \left(\int_0^1 (1-s)^{\alpha p} u_{n,x}(s)ds\right)^{1/p} \left(\int_1^\infty s^{-(\alpha+1)p'} v_{n,x}(s)^{1-p'} ds\right)^{1/p'}$$

and

$$\bar{D}_2^* = \sup_{x\in\mathbb{R}^n} \left(\int_0^1 u_{n,x}(s)ds\right)^{1/p} \left(\int_1^\infty (s-1)^{\alpha p'} s^{-(\alpha+1)p'} v_{n,x}(s)^{1-p'} ds\right)^{1/p'} .$$

Proof. (a) If $x \in \mathbb{R}^n$, then in polar coordinates $x = s\sigma$, where $s > 0$ and $\sigma \in \Sigma_{n-1}$, the unit n-sphere in $\mathbb{R}^n$. For such s and σ, write $u_{n,\sigma}(s) = s^{n-1}u(s\sigma)$,

$v_{n,\sigma} = s^{n-1}v(s\sigma)$, then (3.1) may be written in the form

$$\begin{aligned}\|\bar{I}_\alpha f\|_{p,u} &= \left\{\int_{\Sigma_{n-1}} d\sigma \int_0^\infty u_{n,\sigma}(s)\Big|(\alpha+1)\int_0^1 (1-t)^\alpha f(s\sigma t)dt\Big|^p ds\right\}^{1/p} \\ &= \left\{\int_{\Sigma_{n-1}} d\sigma \int_0^\infty u_{n,\sigma}(s)\Big|\frac{\alpha+1}{s^{\alpha+1}}\int_0^s (s-y)^\alpha f(y\sigma)dy\Big|^p ds\right\}^{1/p} \quad (y=st) \\ &\le C_0\left\{\int_{\Sigma_{n-1}} d\sigma \int_0^\infty v_{n,\sigma}(s)|f(s\sigma)|^p ds\right\}^{1/p} = C_0\|f\|_{p,v}\ . \qquad (3.3)\end{aligned}$$

But by Theorem 2.1(a) with $q = p$ and u, v replaced by $u_{n,\sigma}$, $v_{n,\sigma}$, respectively, (3.3) is satisfied if

$$\sup_{t>0}\left(\int_0^\infty (s-t)^{\alpha p} s^{-(\alpha+1)p} u_{n,\sigma}(s)ds\right)^{1/p}\left(\int_0^1 v_{n,\sigma}(s)^{1-p'}ds\right)^{1/p'} < \infty \quad (3.4)$$

and

$$\sup_{t>0}\left(\int_t^\infty s^{-(\alpha+1)p} u_{n,\sigma}(s)ds\right)^{1/p}\left(\int_0^t (t-s)^{\alpha p'} v_{n,\sigma}(s)^{1-p'}ds\right)^{1/p'} < \infty\ . \quad (3.5)$$

But with $s = rt$, $ds = tdr$ it is easily seen that these integral products are bounded if $\bar{D} < \infty$.

Conversely, if (3.1), or equivalently (3.3) is satisfied for all $f \in L_v^p(\mathbb{R}^n)$, then specifically with $f(s\sigma) = \varphi(\sigma)^{1/p}g(s)$, $s > 0$, $\sigma \in \Sigma_{n-1}$. That is,

$$\begin{aligned}&\int_{\Sigma_{n-1}} d\sigma \int_0^\infty u_{n,\sigma}(s)\Big|(\alpha+1)\int_0^1 (1-t)^\alpha g(st)\varphi(\sigma)^{1/p}dt\Big|^p ds \\ &\qquad \le C_0^p \int_{\Sigma_{n-1}} d\sigma \int_0^\infty v_{n,\sigma}(s)|g(s)|^p\varphi(\sigma)ds\end{aligned}$$

is satisfied for all measurable $\varphi \ge 0$ and hence for a.e. $\sigma \in \Sigma_{n-1}$

$$\int_0^\infty u_{n,\sigma}(s)|(I_\alpha g)(s)|^p ds \le C_0^p \int_0^\infty v_{n,\sigma}(s)|g(s)|^p ds$$

holds. But then by Theorem 2.1(a) with $q = p$ and u, v replaced by $u_{n,\sigma}, v_{n,\sigma}$, respectively, (3.4) and (3.5) are satisfied. But then a change of variable shows that $\bar{D}_1 < \infty$ and $\bar{D}_2 < \infty$. This proves the first part of the theorem.

Part (b) follows in exactly the same way. We omit the details.

We note here that Theorem 3.1(a) with $\alpha = 0$ was proved in [5] for the index range $0 < q \le p$, $p > 1$. The argument given here is essentially the same. In the case $\alpha > 0$, the index range of Theorem 3.1 can also be extended to $0 < q \le p$, $p > 1$, provided the results of Theorem 2.1 are suitably extended.

Corollary 2.1 shows that both I_α and J_α, $\alpha \geq 0$, are bounded on L^p_w, if $w \in A_p$. This result also holds in the n-dimensional case for $\bar{I}_\alpha$ and $\bar{J}_\alpha$ provided a suitable extension of the A_p weight class is given:

Definition 3.2 *(a) Let w be a weight function on $\mathbb{R}^n$. If $x \in \mathbb{R}^n$, then in polar coordinates $x = t\sigma$, $t > 0$, $\sigma \in \Sigma_{n-1}$. For fixed $\sigma \in \Sigma_{n-1}$ write $w_{n,\sigma}(t) = t^{n-1}w(t\sigma)$. $w \in A_{p,\Sigma_{n-1}}$, $1 < p < \infty$ if $w_{n,\sigma} \in A_p$, that is, for all $t > 0$ and any $\sigma \in \Sigma_{n-1}$, there is a constant $C > 0$, such that*

$$\left(\int_0^t w_{n,\sigma}(s)ds\right)^{1/p}\left(\int_0^t w_{n,\sigma}(s)^{1-p'}ds\right)^{1/p'} \leq Ct \tag{3.6}$$

holds.

(b) $w \in B_{p,\Sigma_{n-1}}$, $1 < p < \infty$, if $w_{n,\sigma} \in B_p$, that is,

$$\int_t^\infty \frac{w_{n,\sigma(s)}}{s^p}\,ds \leq Ct^{-p}\int_0^t w_{n,p}(s)ds \tag{3.7}$$

is satisfied.

As in Remark 2.1, $A_{p,\Sigma_{n-1}} \subset B_{p,\Sigma_{n-1}}$ and $w \in A_{p,\Sigma_{n-1}}$, if and only if $w^{1-p'} \in A_{p',\Sigma_{n-1}}$. It follows as before that $w \in A_{\Sigma_{n-1}}$ implies $w^{1-p'} \in B_{p',\Sigma_{n-1}}$.

Corollary 3.1 *If $w \in A_{p,\Sigma_{n-1}}$, $1 < p < \infty$, then $\bar{I}_\alpha$, $\bar{J}_\alpha$, $\alpha \geq 0$ are bounded on $L^p_w(\mathbb{R}^n)$.*

Proof. By Theorem 3.1 it suffices to show that both $\bar{D}$ and $\bar{D}^*$ are finite for $u = v \equiv w$. But since the change of variable $s = tr$ and $x = t\sigma$, $t > 0$, $\sigma \in \Sigma_{n-1}$ shows that (3.6) and (3.7) take the forms

$$\sup_{x\in\mathbb{R}^n}\left(\int_0^t w_{n,x}(r)dr\right)^{1/p}\left(\int_0^1 w_{n,x}(r)^{1-p'}dr\right)^{1/p'} \leq C$$

and

$$\int_0^\infty \frac{w_{n,x}(r)dr}{r^p} \leq C\int_0^1 w_{n,x}(r)dr$$

respectively, this follows from simple estimates.

Corollary 3.2 *If $f \geq 0$ and $w \in A_{p,\Sigma_{n-1}}$, $1 < p < \infty$, then*

$$\|\bar{I}_\alpha f\|_{p,w} \approx \|\bar{J}_\alpha f\|_{p,w}$$

for $\alpha \in \mathbb{N}$.

Proof. If $x = s\sigma$, $s > 0$, $\sigma \in \Sigma_{n-1}$, then the change of variable $r = st$ shows that

$$(\bar{I}_\alpha f)(x) = (\alpha+1)\int_0^1 (1-t)^\alpha f(s\sigma t)dt = \frac{(\alpha+1)}{s^{\alpha+1}}\int_0^s (s-r)^\alpha f(\sigma r)dr$$

and

$$(\bar{J}_\alpha f)(x) = (\alpha+1)\int_1^\infty (t-1)^\alpha t^{-\alpha-1} f(s\sigma t)dt = (\alpha+1)\int_s^\infty (r-s)^\alpha r^{-\alpha-1} f(r\sigma)dr\ .$$

The result then follows from Theorem 2.2 with f replaced by $f(t\sigma)$ and w by $w_{n,\sigma}$.

4 Applications

In this section we establish the equivalence between weighted norm inequalities of the operators $\bar{I}_\alpha$, $\bar{J}_\alpha$ and certain differential inequalities in weighted Lebesgue spaces. These results are motivated by corresponding work of G. Sinnamon [5] in the case $\alpha = 0$.

We remind the reader that if $x, y \in \mathbb{R}^n$, then

$$x \cdot y = (x_1, \ldots, x_n)\cdot(y_1, \ldots, y_n) = \sum_{i=1}^n x_i y_i$$

and the gradient of a function g on $\mathbb{R}^n$ is

$$\nabla g(x) = \left(\frac{\partial g(x)}{\partial x_1}, \ldots, \frac{\partial g(x)}{\partial x_n}\right),$$

First we recall the following:

Theorem 4.1 *(Sinnamon [5]) Suppose $f, g \in C_0^1(\mathbb{R}^n)$ u, v weight functions on $\mathbb{R}^n$ and $1 < p < \infty$. If any one estimate of part (a), respectively, part (b) is satisfied, then so are the other two.*

(a)

$$\|\bar{I}_0 f\|_{p,u} \le C\|f\|_{p,v} \tag{4.1}$$

$$\sup_{x\in\mathbb{R}^n}\left(\int_1^\infty s^{-p}[s^{n-1}u(sx)]ds\right)^{1/p}\left(\int_0^1 [s^{n-1}v(sx)]^{1/p'}ds\right)^{1/p'} < \infty \tag{4.2}$$

$$\left(\int_{\mathbb{R}^n} |g(x)|^p u(x)dx\right)^{1/p} \le C\left(\int_{\mathbb{R}^n} |g(x) + x\cdot\nabla g(x)|^p v(x)dx\right)^{1/p}. \tag{4.3}$$

(b)

$$\|\bar{J}_0 f\|_{p,u} \le C\|f\|_{p,v} \tag{4.4}$$

$$\sup_{x\in\mathbb{R}^n}\left(\int_0^1 [s^{n-1}u(sx)]ds\right)^{1/p}\left(\int_1^\infty s^{-p'}[s^{n-1}v(sx)]^{1-p'}ds\right)^{1/p'} \tag{4.5}$$

$$\left(\int_{\mathbb{R}^n}|g(x)|^p u(x)dx\right)^{1/p} \le C\left(\int_{\mathbb{R}^n}|x\cdot\nabla g(x)|^p v(x)dx\right)^{1/p}. \tag{4.6}$$

Proof. The equivalence of (4.1) and (**??**) as well as (4.4) and (**??**) is the content of Theorem 3.1 with $\alpha = 0$. The equivalence of (4.4) and (4.6) was proved in [7, Theorem 4.1] and (4.1) and (4.3) is proved in a similar way. We give the details here for completeness.

If $\lambda > 0$, then by Definition 3.1 with $\alpha = 0$,

$$(\bar{I}_0 f)(\lambda x) = \int_0^1 f(x\lambda t)dt = \frac{1}{\lambda}f(xs)ds\ , \quad (s = \lambda t)$$

so that

$$f(x\lambda) = \frac{d}{d\lambda}[\lambda(\bar{I}_0 f)(\lambda x)] = \lambda x\cdot\nabla(\bar{I}_0 f)(\lambda x) + (\bar{I}_0 f)(\lambda x)\ ,$$

and as $\lambda \to 1$, $f(x) = x\cdot\nabla(\bar{I}_0 f)(x) + (\bar{I}_0 f)(x)$. Conversely, if $g \in C_0^1(\mathbb{R}^n)$ and $f(x) = x\cdot\nabla g(x) + g(x)$, then

$$\begin{aligned}(\bar{I}_0 f)(x) &= \int_0^1 f(xt)dt = \int_0^1 [tx\cdot\nabla g(tx) + t(tx)]dt \\ &= \int_0^1 \frac{d}{dt}[tg(tx)]dt = g(x)\ .\end{aligned}$$

Hence (4.1) and (4.3) are equivalent.

Note that for $w \in A_{p,\Sigma_{n-1}}$, $1 < p < \infty$, Corollary 3.1 shows that $\bar{I}_0$ and $\bar{J}_0$ are bounded on $L^p_w(\mathbb{R}^n$. Moreover, by Corollary 3.2 $\bar{I}_0 f$ and $\bar{J}_0 f$ are comparable whenever $f \ge 0$. Thus we obtain from Theorem 4.1.

Corollary 4.1 *Suppose $w \in A_{p,\Sigma_{n-1}}$, $1 < p < \infty$, $f, g \in C_0^1(\mathbb{R}^n)$ where $f \ge 0$ and $g \ge 0$. Then with $u = v \equiv w$, (4.1), (4.3), (4.4) and (4.6) are equivalent.*

Considering the operators $\bar{I}_1$ and $\bar{J}_1$ one obtains in a similar way the following:

Theorem 4.2 *Suppose $f, g \in C_0^2(\mathbb{R}^n)$ u, v weight functions on $\mathbb{R}^n$ and $1 < p < \infty$. If any one of the estimates (4.7), (4.8, (4.9) of part (a), respectively, (4.10), (4.11), (4.12) of part (b) re satisfied, then so are the other two.*

(a)

$$\|\bar{I}_1 f\|_{p,u} \leq C\|f\|_{p,v} \tag{4.7}$$

$$\sup_{x\in\mathbb{R}^n}\left(\int_1^\infty (s-1)^p s^{-2p}[s^{n-1}u(sx)]ds\right)^{1/p}\left(\int_0^1 [s^{n-1}v(sx)]^{1-p'}ds\right)^{1/p'} < \infty$$

and (4.8)

$$\sup_{x\in\mathbb{R}^n}\left(\int_1^\infty s^{-2p}[s^{n-1}u(sx)]ds\right)^{1/p}\left(\int_0^1 (1-s)^p[s^{n-1}v(sx)]^{1-p'}ds\right)^{1/p'} < \infty$$

$$\begin{aligned}&\left(\int_{\mathbb{R}^n}|g(x)|^p u(x)dx\right)^{1/p}\\ &\leq \; C\left(\int_{\mathbb{R}^n}\left|g(x)+2^\alpha x\cdot\nabla g(x)+\frac{1}{2}\sum_{i,j=1}^n x_i x_j\frac{\partial^2 g(x)}{\partial x_i\partial x_j}\right|^p v(x)dx\right)^{1/p}\end{aligned} \tag{4.9}$$

(b)

$$\|\bar{J}_1 f\|_{p,u} \leq C\|f\|_{p,v} \tag{4.10}$$

$$\sup_{x\in\mathbb{R}^n}\left(\int_0^1 (1-s)^p[s^{n-1}u(sx)]ds\right)^{1/p}\left(\int_1^\infty s^{-2p'}[s^{n-1}v(sx)]^{1-p'}ds\right)^{1/p'} < \infty$$

and (4.11)

$$\sup_{x\in\mathbb{R}^n}\left(\int_0^1 s^{-2p}[s^{n-1}u(sx)]ds\right)^{1/p}\left(\int_1^\infty (s-1)^{p'}s^{-2p'}[s^{n-1}v(sx)]^{1-p'}ds\right)^{1/p'} < \infty$$

$$\begin{aligned}&\left(\int_{\mathbb{R}^n}|g(x)|^p u(x)dx\right)^{1/p}\\ &\leq \; C\left(\int_{\mathbb{R}^n}\left|\sum_{i,j=1}^n x_i x_j\frac{\partial^2 g(x)}{\partial x_i\partial x_j}\right|^p v(x)dx\right)^{1/p}.\end{aligned} \tag{4.12}$$

Proof. The equivalent of (4.7) and (4.8), as well as that of (4.10) and (4.11) is given in Theorem 3.1(a), respectively (b).

To establish the equivalence of (4.7) and (4.9), observe that for $\lambda > 0$, Definition 3.1 shows that

$$(\bar{I}_1 f)(x\lambda) = 2\int_0^1 (1-t) f(x\lambda t)dt = \frac{2}{\lambda^2}\int_0^\lambda (\lambda - s) f(xs)ds \quad (t\lambda = s).$$

Applying the chain rule twice yields

$$\begin{aligned} 2f(x\lambda) &= \frac{d^2}{d\lambda^2}[\lambda^2(\bar{I}_1 f)(x\lambda)] = \frac{d}{d\lambda}[2\lambda(\bar{I}_1 f)(x\lambda) + \lambda^2 x\cdot\nabla(\bar{I}_1 f)(x\lambda)] \\ &= 2(\bar{I}_1 f)(x\lambda) + 2\lambda x\cdot\nabla(\bar{I}_1 f)(x\lambda) + 2\lambda x\cdot\nabla(\bar{I}_1 f)(x\lambda) \\ &\quad + \lambda^2\sum_{i,j=1}^n x_i x_j \frac{\partial^2(\bar{I}_1 f)(x\lambda)}{\partial x_i \partial x_j} \end{aligned}$$

and as $\lambda \to 1$,

$$2f(x) = 2(\bar{I}_1 f)(x) + 4x\cdot\nabla(\bar{I}f)(x) + \sum_{i,j=1}^n x_i x_j \frac{\partial^2(\bar{I}_1 f)(x)}{\partial x_i \partial x_j} .$$

Conversely, if

$$2f(x) = 3g(x) + 4x\cdot\nabla g(x) + \sum_{i,j=1}^n x_i x_j \frac{\partial^2 g(x)}{\partial x_i \partial x_j} ,$$

then

$$\begin{aligned} (\bar{I}_1 f)(x) &= 2\int_0^1 (1-t) f(xt)dt \\ &= \int_0^1 (1-t)\left[2g(xt) + 4tx\cdot\nabla g(xt) + t^2\sum_{i,j=1}^n x_i x_j \frac{\partial^2 g(xt)}{\partial x_i \partial x_j}\right] dt \\ &= \int_0^1 (1-t)\frac{d^2}{dt^2}[t^2 g(xt)]dt = \int_0^1 \frac{d}{dt}[t^2 g(xt)]dt = g(x) \end{aligned}$$

It follows therefore that (4.7) and (4.9) are equivalent.

The equivalence of (4.10) and (4.12) is obtained in the same way and given for completeness only.

Again by Definition 3.1 and a change of variable

$$(J_1 f)(\lambda x) = 2\int_\lambda^\infty (s-\lambda)s^{-2} f(sx)ds ,$$

so that differentiation and applications of the chain rule yields

$$\begin{aligned} 2\lambda^{-2} f(x\lambda) &= \frac{d^2}{d\lambda^2}(\bar{J}_1 f)(\lambda x) = \frac{d}{d\lambda}[x \cdot \nabla(\bar{J}_1 f)(\lambda x)] \\ &= \sum_{i,j=1}^{n} x_i x_j \frac{\partial^2 (\bar{J}_1 f)(\lambda x)}{\partial x_i \partial x_j)} . \end{aligned}$$

Hence as $\lambda \to 1$, $2f(x) = \sum_{i,j=1}^n x_i x_j \frac{\partial^2 (\bar{J} f)(\lambda x)}{\partial x_i \partial x_j}$.

Conversely, if $2f(x) = \sum_{i,j=1}^n x_i x_j \frac{\partial^2 g(x)}{\partial x_i \partial x_j}$, then

$$\begin{aligned} (\bar{J}_1 f)(x) &= 2\int_1^\infty (t-1)t^2 f(xt)dt = \int_1^\infty (t-1)\left[\sum_{i,j=1}^n x_i x_j \frac{\partial^2 g(xt)}{\partial x_i \partial x_j}\right] dt \\ &= \int_1^\infty (t-1)\frac{d^2}{dt^2}[g(xt)]dt = -\int_1^\infty \frac{d}{dt}[g(xt)]dt = g(x), \end{aligned}$$

and the result follows.

If $w \in A_{p,\Sigma_{n-1}}$, $1 < p < \infty$, then by Corollary 3.1 $\bar{I}_\alpha$ and $\bar{J}_\alpha$, $\alpha \geq 0$, are bounded on $L^p_w(\mathbb{R}^n)$. This and Theorem 4.2 yields

Corollary 4.2 *If $f, g \in C_0^2(\mathbb{R}^n)$ and $w \in A_{p,\Sigma_{n-1}}$, $1 < p < \infty$, then with $u = v \equiv w$, (4.7) and (4.9), respectively, (4.10) and (4.12) are equivalent.*

Observe that by Schwarz's inequality

$$\left|\sum_{i,j=1}^n x_i x_j \frac{\partial^2 g(x)}{\partial x_i \partial x_j}\right| \leq |x|^2 \left(\sum_{i,j=1}^n \left|\frac{\partial^2 g(x)}{\partial x_i \partial x_j}\right|^2\right)^{1/2}$$

so that for $w \in A_{p,\Sigma_{n-1}}$, $1 < p < \infty$

$$\left(\int_{\mathbb{R}^n} |g(x)|^p w(x)dx\right)^{1/p} \leq C\left(\int_{\mathbb{R}^n} |x|^{2p}\left(\sum_{i,j=1}^n \left|\frac{\partial g(x)}{\partial x_i \partial x_j}\right|^2\right)^{p/2} w(x)dx\right)^{1/p} \quad (4.13)$$

Specifically, if $w(x) = |x|^{-2p}$ then $w \in A_{p,\Sigma_{n-1}}$, if $\frac{n}{3} < p < \frac{n}{2}$, and (4.13) is satisfied with this weight if $p \in (n/3, n/2)$, $n > 3$.

It is now clear how higher order differential inequalities follow from the boundedness of the operators $\bar{I}_\alpha$, $\bar{J}_\alpha$, $\alpha \in \mathbb{N}$ given in Theorem 3.1, or Corollary 3.1. For example, if $\alpha = 2$ and $w \in A_{p,\Sigma_{n-1}}$, $1 < p < \infty$, then for $f, g \in C_0^3(\mathbb{R}^n)$

$$\|\bar{J}_2 f\|_{p,w} \leq C\|f\|_{p,w}$$

and

$$\left(\int_{\mathbb{R}^n} |g(x)|^p w(x)dx\right)^{1/p} \leq C\left(\int_{\mathbb{R}^n}\left|\sum_{i,j,k=1}^n x_i x_j x_k \frac{\partial^3 g(x)}{\partial x_i \partial x_j \partial x_k}\right|^p w(x)dx\right)^{1/p}$$

are equivalent.

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