Int. J. Contemp. Math. Sci., Vol. 2, 2007, no. 14, 673 - 677

A Remark on $C^{\star}$-Dynamical Systems in Hilbert $C^{\star}$-modules

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Abstract

In this paper we consider C_0-group of unitary operators on a Hilbert $C^{\star}$-module E. In particular we show that if $A \subseteq L(E)$ is a $C^{\star}$-algebra including $K(E)$ and there is $x \in E$ with $< x, x >= 1$ and if α_t is a $C^{\star}$-dynamics on A with generator δ and $\theta_{x,x} \in D(\delta)$, then there is $C^{\star}$-dynamics $\alpha_{x,t}$ such that $\alpha_{x,t}(\theta_{x,x}) = \theta_{x,x} \quad t \in \mathcal{R}$, and there is a C_0-group $u_{x,t}$ of unitaries in E such that $\alpha_{x,t}(a) = u_{x,t} a u^{\star}_{x,t}$ for $a \in K(E)$

Mathematics Subject Classification: 37B99, 47D03, 46C99

Keywords: $C^{\star}$-dynamical system ,C_0-group of linear Operators, Hilbert $C^{\star}$- module, Unitary Operator

1. Introduction

A one parameter family $T : \mathcal{R} \to B(X), T = T(t), t \in \mathcal{R}$ of bounded operators on a Banach space X is called a one parameter group if it satisfies:

i) $T(0) = 1$
ii) $T(s+t) = T(s)T(t) \quad t, s \in R$.
Moreover, $T(t)$ is called C_0-group if iii) For each $x \in X$ the map $t \longrightarrow T(t)x$ from $\mathcal{R}$ to X is continuous with respect to norm topology of X.
We define the infinitesimal generator of the one parameter group $T(t)$ by

$$\delta x = \lim_{t \to 0} t^{-1}[T(t)x - x].$$

Where the domain $D(\delta)$ of δ is the set of all $x \in X$ such that the limit exists.
Let A be a $\star$- algebra. An automorphism on A is an invertible linear operator $\alpha: \; A \longrightarrow A$ such that $\alpha(ab) = \alpha(a)\alpha(b)$ and $\alpha(a^\star) = (\alpha(a))^\star$. The triple (A, G, α) is called a strongly continuous $C^\star$-dynamical system (or briefly $C^\star$-dynamical system) if:
i) A is $C^\star$-algebra and G is a locally compact topological group
ii) $\alpha : G \longrightarrow aut(A)$ is a group homomorphism of G into the group of $\star$-automorphism of A, $g \longrightarrow \alpha(g)$ is a continuous mapping of G into $aut(A)$,where the topology of $aut(A)$ is strong operator topology on $aut(A)$ as a subspace of $B(A)$,the algebra of bounded operators on A.
In the case that the homomorphism $\alpha : G \longrightarrow B(A)$ is continuous with respect to norm topology of $B(A), (A, G, \alpha)$ is called uniformly continuous $C^\star$-dynamical system. Let α_t be a group of $\star$-automorphisms on $C^\star$-algebra A and δ be the generator of α_t,then δ is a $\star$-derivation. Converesly if δ is a bounded $\star$-derivation on $C^\star$-algebra A,δ induces a uniformly continuous group of $\star$-automorphisms,$\alpha_t = e^{t\delta}$ on A.If h is a self-adjoint element in a $C^\star$-algebra A then by stone theorem [2;(1.10.8)] ih is the infinitesimal generator of uniformly continuous group u_t of unitaries in A, such that $u_t = e^{ith}$ and $\delta(a) = [ih, a] = i[h, a]$,where we write $[a, b]$ for $ab - ba.\delta$ is an inner derivation on A which is the infinitesimal generator of the uniformly continuous group of inner $\star$-automorphism:

$$\alpha_t(a) = u_t a u_t^\star = e^{ith} a e^{-ith}$$

(see [3] for more result on $C^\star$-dynamical system).Suppose A is a $C^\star$-algebra. Let E be a complex linear space which is a left A-module and $\lambda(ax) = (\lambda a)x = a(\lambda x)$ where $\lambda \in \mathcal{C}$, $a \in A$ and $x \in E$. The space E is called a pre-Hilbert A-module if there exists an (A-valued) inner product $<,>$: $E \times E \to A$ such that for every $x, y \in E, \lambda \in \mathcal{C}$ and $a \in A$, we have:

$(i) < x, x > \geq 0$
$(ii) < x, x >= 0 \;\; iff \;\; x = 0$

$(iii) < x+\lambda y, z >=< x, z > +\lambda < y, z >$
$(iv) < x, y >=< y, x >^{\star}$
$(v) < ax, y >= a < x, y >$

A pre-Hilbert A-module E is called a Hilbert A-module or Hilbert $C^{\star}$-module over A, if it is complete with respect to the norm $\|x\| = \| < x, x > \|^{\frac{1}{2}}$. For example if A is a $C^{\star}$-algebra, then A with its product as the usual action is left A-module. In addition if A equipped with the inner product $< a, b >= ab^{\star}$ then it is a Hilbert A-module.

Suppose that E, F are Hilbert $C^{\star}$-modules. We define $L(E, F)$ to be the set of all maps $t : E \longrightarrow F$ for which there is a map $t^{\star} : F \longrightarrow E$ such that $< tx, y >=< x, t^{\star}y > \quad (x \in E,\ y \in F)$. It is easy to see that t must be A-linear and bounded ([1], P.8).We call $L(E, F)$ the set of adjointable maps from E to F. Thus every element of $L(E, F)$ is a bounded A-linear map.In particular, $L(E, E)$ which we abbreviate to $L(E)$ is a $\star$- algebra. Let E and F be Hilbert $C^{\star}$-modules. For $x \in E$ and $y \in F$, define $\theta_{\ x,y} :\ F \longrightarrow E$ by $\theta_{x,y}(z) =< z, y > x \quad (z \in F)$. It is easy to check that $\theta_{\ x,y} \in L(E, F)$ with $(\theta_{x,y})^{\star} = \theta_{y,x}$ and also that the following relations hold: (where G is Hilbert $C^{\star}$-module)
$t\theta_{x,y} = \theta_{tx,y} \qquad (t \in L(E, G))$
$\theta_{x,y}s = \theta_{x,s^{\star}y} \qquad (s \in L(G, F))$.
We denote by $K(F, E)$ the closed linear subspace of $L(F, E)$ spanned by $\{\theta_{x,y} \ \ x \in E, y \in F\}$ and we write $K(E)$ for $K(E, E)$. We also have $1x = x$ if A has an identity 1 and also $\overline{K(E)E} = E$ ([1], P.18). An operator $u \in L(E, F)$ is said to be unitary if $u^{\star}u = 1_E$ and $uu^{\star} = 1_F$ and called self adjoint if $u^{\star} = u$. In this paper we consider Hilbert A-module E over unital $C^{\star}$-algebra A. (see [1] for more details on Hilbert $C^{\star}$-module).

2. The main results

Theorem.1 Let u_t be a C_0-group of unitary operators on a Hilbert $C^{\star}$-module E.Then $\alpha_t(a) = u_t a u_t^{\star}$ $(a \in K(E))$ is a C_0-group of automorphism on $K(E)$.[2]

Theorem.2 Let A be a $C^{\star}$-algebra and $K(E) \subseteq A \subseteq L(E)$ and α_t a C_0 -group of $\star$-automorphisms on A such that there is $x \in E$ with $< x, x >= 1$ and $\alpha_t(\theta_{x,x}) = \theta_{x,x} \quad t \in \mathcal{R}$, then there is a C_0- group u_t of unitaries in $L(E)$ such that $\alpha_t(a) = u_t a u_t^{\star}$.[2]

Lemma 3 Let A be a $C^\star$-algebra on Hilbert $C^\star$-module E which includes $K(E)$ as a $C^\star$-subalgebra.Also let α_t be a C_0-group of automorphisms on A with generator δ.If there is $x \in E$ such that $< x, x >= 1$ and $\theta_{x,x} \in D(\delta)$,then there is a bounded $\star$-derivation δ_x such that $(\delta + \delta_x)(\theta_{x,x}) = 0$ and it generates a C_0-group of automorphisms on A.

proof : We define $h_x = i[\delta(\theta_{x,x}), \theta_{x,x}]$ and $\delta_x(a) = i[h_x, a]$. h_x is self-adjoint and δ_x is a bounded derivation on A.Also $(\delta + \delta_x)(\theta_{x,x}) = \delta(\theta_{x,x}) + i[h_x, \theta_{x,x}] = \delta(\theta_{x,x}) + i(h_x\theta_{x,x} - \theta_{x,x}h_x) = \delta(\theta_{x,x}) - (\delta(\theta_{x,x})\theta_{x,x} - \theta_{x,x}\delta(\theta_{x,x})\theta_{x,x} - \theta_{x,x}\delta(\theta_{x,x})\theta_{x,x} + \theta_{x,x}\delta(\theta_{x,x})) = \delta(\theta_{x,x}) - (\delta(\theta_{x,x})\theta_{x,x} + \theta_{x,x}\delta(\theta_{x,x})) = \delta(\theta_{x,x}) - \delta(\theta_{x,x}^2) = 0$.Because $\theta_{x,x}$ is projection and $\delta(\theta_{x,x}) = \delta(\theta_{x,x}^2) = \delta(\theta_{x,x})\theta_{x,x} + \theta_{x,x}\delta(\theta_{x,x})$. Now δ is a generator and δ_x is bounded by perturbation theorem [2;(3.3.2)] $\delta + \delta_x$ generates a C_0-group $\alpha_{x,t}$ of operators on A.But $\delta+\delta_x$ is a derivation then for each t,$\alpha_{x,t}$ is an automorphism .Therfore $\alpha_{x,t}$ is also a C_0-group of automorphisms on A.In addition,since $\delta + \delta_x$ is a $\star$-derivation $\alpha_{x,t}$ is a $\star$-automorphism for each $t \in \mathcal{R}$ $\square$.

Theorem 4 Let A be a $C^\star$-algebra acting on Hilbert $C^\star$-module E and If α_t is a $C^\star$-dynamics on A with generator δ,and there is $x \in E$ such that $< x, x >= 1$ and $\theta_{x,x} \in D(\delta)$, then there is $C^\star$-dynamics $\alpha_{x,t}$ such $\alpha_{x,t}(\theta_{x,x}) = \theta_{x,x} \quad t \in \mathcal{R}$, and there is a C_0-group $u_{x,t}$ of unitaries in E such that $\alpha_{x,t}(a) = u_{x,t}au^\star_{x,t}$ for $a \in D(\delta) \cap K(E)$.

Proof : By lemma 3 there is a bounded $\star$-derivation δ_x where $\delta_x(a) = i[h_x, a]$ such that $(\delta + \delta_x)(\theta_{x,x}) = 0$. So if $\alpha_{x,t}$ is $C^\star$-dynamics generated by $\delta + \delta_x$,then $\alpha_{x,t}(\theta_{x,x}) = \theta_{x,x}$ for every $t \in \mathcal{R}$. This means that $\theta_{x,x}$ is invariant for $\alpha_{x,t}$ so by theorem 2 there is a C_0-group $u_{x,t}$ of unitaries such that $\alpha_{x,t}(a) = u_{x,t}au^\star_{x,t}$.

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Received: October 5, 2006