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On the Existence of the Solution of an Optimal Shape Design Problem Governed by Full Navier-Stokes Equations

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Abstract

This paper deals with the existence question in an optimal shape design(OSD) problem governed by full Navier-Stokes equations. Here designing the shape of a wing while a part of deformation tensor is to be reduced, is imagined as the determining the shape of a bump on a part of the boundary of a virtual channel in which the two-dimensional compressible flow of a viscous fluid passes. To prove the existence of an optimal solution of the mentioned OSD problem, we use a general method in the calculus of variations. This method is represented in the proof of main theorem, stated in the last section.

Mathematics Subject Classification: 49J20, 76D05

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1 Introduction

Optimal shape design (OSD) problem has been desirable and ancient subject that freshly, due to commercial and economical attention, has received wealthy regards in the engineering studies and specially in the mathematical

community because of accomplishing the OSD problems with complex equations broadly.
Fluid flow models associated with the Navier-Stokes equations have become a matter of deep study in the mathematical investigations. Although the range of mathematical issues in the OSD problems is wide, but the problems governed by various cases of the Navier-Stokes equations, for instance, compressible or incompressible, viscous or inviscid, are in their beginning and infancy.
At present a few mathematical literature in such studies are available, see [5], [2], [3] and [12].
In this paper we consider a particular OSD problem in which minimization of viscous drag is studied through shape modification.
Governing this OSD problem by full Navier-Stokes equations, i.e, compressible viscous Navier-Stokes equations, is our main interest in this paper.
Our aim here is to investigate the existence of an special drag-minimizing shape.
In sequel we state an OSD problem of minimum wing drag which is extensively watchfulness in aerospace fields.

2 The OSD problem

We are going to design the shape of a wing such that the drag force due to viscosity, is to be reduced. To study this problem we consider the shape of a wing in a fixed vertical slit as a bump included in a virtual channel in which the two-dimensional compressible flow of a viscous fluid passing through. See Figure 1

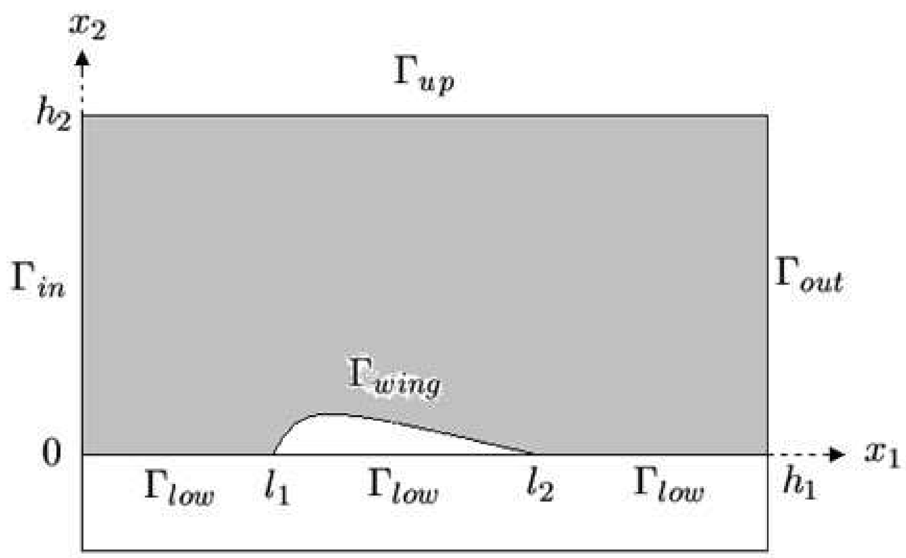

Figure 1. The geometry of problem

The shape of wing is separated into the two parts: the fixed side Γ_{low} and the moving side to be designed, say Γ_{wing}.
The arc Γ_{wing}, bump of virtual channel, is assumed to be

$$\Gamma_{wing}(\alpha) = \{(x_1, x_2) \in [l_1, l_2] \times [0, h_2]; x_2 = \alpha(x_1), \alpha : [l_1, l_2] \longrightarrow [0, h_2)\},$$

this arc is to be designed in an optimization process.
We introduce the domain $\Omega(\alpha)$, shadowed region in Figure 1., is made up of three parts consisting of the two fixed rectangles in the left and right side of an area with an unknown boundary, the graph of the curve α. Let $\Gamma(\alpha)$ be $\partial\Omega(\alpha)$.
Now we consider the following boundary value problem for the nondimensionalized barotropic compressible viscous Navier-Stokes equations (see [7], [11]):

$$-\mu\triangle\mathbf{u} - \nu\nabla div\mathbf{u} + \varrho(p)(\mathbf{u}.\nabla)\mathbf{u} + \nabla p = 0 \qquad in\ \Omega(\alpha) \quad (1)$$

$$div(\varrho\mathbf{u}) = 0 \qquad in\ \Omega(\alpha) \quad (2)$$

$$\mathbf{u} = \mathbf{u}_0 \qquad on\ \Gamma(\alpha) \quad (3)$$

$$p = p_0 \qquad on\ \Gamma_{in}(\alpha), \quad (4)$$

where $\mathbf{u} = (u_1, u_2)$ is the velocity vector, p is the pressure, $\varrho = \varrho(p)$ is the given density, μ and ν are the coefficients of viscosity which satisfy the thermodynamic restrictions $\mu > 0$, $\nu + \mu > 0$. The initial functions $\mathbf{u}_0 = (u_{0,1}, u_{0,2})$ and p_0 are given smooth on the closure of $\Omega(\alpha)$.
We intend to transform $(1) - (4)$ into a homogeneous boundary value problem, so inserting proper functions $\tilde{\mathbf{u}} = \mathbf{u} - \mathbf{u}_0$ and $\tilde{p} = p - p_0$ to $(1) - (4)$ and substituting again $\tilde{\mathbf{u}}$ and $\tilde{p}$ by $\mathbf{u}$ and p, results that:

$$-\mu\triangle\mathbf{u} - \nu\nabla div\mathbf{u} + \nabla p + \varrho\{(\mathbf{u} + \mathbf{u}_0).\nabla)\mathbf{u} + (\mathbf{u}.\nabla)\mathbf{u}_0\} = \mathbf{f}_0 \qquad in\ \Omega(\alpha) \quad (5)$$

$$div\mathbf{u} + k[(\mathbf{u} + \mathbf{u}_0).\nabla p + \mathbf{u}.\nabla p_0] = g_0 \qquad in\ \Omega(\alpha) \quad (6)$$

$$\mathbf{u} = 0 \qquad on\ \Gamma(\alpha) \quad (7)$$

$$p = 0 \qquad on\ \Gamma_{in}(\alpha). \quad (8)$$

Here $\mathbf{f}_0 = \mu\triangle\mathbf{u}_0 + \nu\nabla div\mathbf{u}_0 - \nabla p_0 - \varrho(\mathbf{u}_0.\nabla)\mathbf{u}_0$, $\varrho = \varrho(p + p_0)$, $g_0 = -div\mathbf{u}_0 - k\mathbf{u}_0$ and $k = \frac{\dot{\varrho}}{\varrho}$ where $\dot{\varrho} = \frac{\partial\varrho}{\partial p}$.
The problem $(5) - (8)$ is rephrased, corresponding to the some linearizations, by the following one:

$$-\mu\triangle\mathbf{u} - \nu\nabla div\mathbf{u} + \varrho(\mathbf{U}.\nabla)\mathbf{u} + \nabla p = \mathbf{f} \qquad in\ \Omega(\alpha) \quad (9)$$

$$div\mathbf{u} + k\mathbf{U}.\nabla p = g \qquad in\ \Omega(\alpha) \quad (10)$$

$$\mathbf{u} = 0 \qquad on\ \Gamma(\alpha) \quad (11)$$

$$p = 0 \qquad on\ \Gamma_{in}(\alpha). \quad (12)$$

Here $\varrho = \varrho(\hat{p}+p_0)$, $k = k(\hat{p}+p_0)$, $\mathbf{U} = \hat{\mathbf{u}}+\mathbf{u}_0$, $\mathbf{f}(\hat{\mathbf{u}},\hat{p}) = \mathbf{f}_0 - \varrho(\hat{\mathbf{u}}.\nabla)\mathbf{u}_0$, and $g(\hat{\mathbf{u}},\hat{p}) = g_0 - k\hat{\mathbf{u}}.\nabla p_0$, where given functions $\hat{\mathbf{u}}$ and $\hat{p}$ are zero on $\Gamma(\alpha)$ and $\Gamma_{in}(\alpha)$, respectively. Let $\mathcal{S}_u^p$ be the set of all $(\mathbf{u}(\alpha), p(\alpha))$ satisfying $(9)-(12)$. According to the function α, it is obvious that determining α, leads to determining the domain $\Omega(\alpha)$. On the other hand as stated by Pironneau in [12], the set of admissible family of functions describing $\Gamma_{wing}(\alpha)$ must not only contain continuous functions because the optimal shape may not exist, therefore we specify $\Gamma_{wing}(\alpha)$ as a set of Lipschitz continuous functions(L.c.f) which are zero in l_1 and l_2, as follows:

$$\mathcal{A}_{ad} = \{\alpha \in \mathcal{C}([l_1, l_2]);\ 0 < \alpha(x_1) < h_2,\ \alpha\ is\ \ L.c.f\}, \tag{13}$$

and we refer to $\mathcal{A}_{ad}$ as the set of admissible controls.
What we are going to minimize, so called, cost functional, is a part of deformation tensor which as pointed out by Mohammadi in [10] is relevant to Navier-Stokes equations.
We want to find $\alpha \in \mathcal{A}_{ad}$ in such way that

$$\mathcal{J}(\alpha, \mathbf{u}(\alpha)) = \|\ \mu(\ \nabla\mathbf{u}(\alpha) + (\nabla\mathbf{u}(\alpha))^T\ ) + (\xi - \frac{2}{3}\mu)\nabla\mathbf{u}(\alpha)\mathcal{I}\ \|_{\odot,\Omega(\alpha)}\ , \tag{14}$$

is minimized. Here $\nu = \xi + \frac{1}{3}\mu$, $\xi \geq 0$ and $\mathcal{I}$ is the identity matrix and $\mathbf{u}(\alpha)$ is a solution of $(9)-(12)$. Also for each arbitrary matrices $A_{n\times n}$ and $B_{n\times n}$, we introduce $A_{n\times n} \odot B_{n\times n} = \sum_{i,j=1}^{n} a_{ij}b_{ij}$, as the matrix multiplication and the inner product $(.,.)_{\odot,\Omega}$ as follows:

$$(A_{n\times n}\ , B_{n\times n})_{\odot,\Omega} = \int_\Omega A_{n\times n} \odot B_{n\times n}\ d\Omega,$$

and consequently the norm $\|.\|_{\odot,\Omega}$ is

$$\|A_{n\times n}\|^2_{\odot,\Omega} = (A_{n\times n}\ , A_{n\times n})_{\odot,\Omega}.$$

Summary our OSD problem is:

$$\min_{\alpha\in\mathcal{A}_{ad}} \mathcal{J}(\alpha, \mathbf{u}(\alpha)) \tag{15}$$

$$s.t.\ \ (\mathbf{u}(\alpha), p(\alpha)) \in \mathcal{S}_u^p. \tag{16}$$

3 Preliminaries and weak variational form

In what follows, we denote the Sobolev function space by $H^{l,p}(\Omega)$, $l \geq 0$, endowed with the norm $\|.\|_{l,p,\Omega}$, and the norm in $L^p(\Omega)$, $1 \leq p \leq \infty$, by $|.|_{p,\Omega}$.

Obviously $\|.\|_{0,2,\Omega} = |.|_{2,\Omega}$.
The Sobolev space for integer $l \geq 0$ is defined as follows:

$$H^{l,p}(\Omega) = \{V \in L^p(\Omega) : \|V\|_{l,p,\Omega} = (\sum_{|\alpha| \leq l} |D^\alpha V|^p_{p,\Omega})^{\frac{1}{p}} < \infty\}.$$

In the case $p = 2$, we mark $H^{l,2} = H^l$ and $\|.\|_{l,2,\Omega} = \|.\|_l$. If $p = \infty$ the norm of L^∞ is showed by $|u|_\infty = ess\ Sup\{|u(x)| : x \in \Omega\}$, and $|u|_{l,\infty} = \sum_{|\alpha|=l} |D^\alpha u|_\infty$, the norm of $H^{l,\infty}$.
$H^{l-\frac{1}{2},p}(\partial\Omega)$, $l \geq 1$ an integer, is the space of traces of elements $u \in H^{l,p}(\Omega)$. The norm in $H^{l-\frac{1}{2},p}(\partial\Omega)$ is indicated by $\|.\|_{l-\frac{1}{2},p,\partial\Omega}$.
We set $H_0^{l,p} = H^{l,p} \bigcap H_0^{1,p}$ where $H_0^{1,p}$ is the space of all functions $u \in H^{1,p}(\Omega)$ such that are zero on $\partial\Omega$. Also $H^{-l,p}$ shows the dual space of $H_0^{l,p'}$ with norm $\|S\|_{-l,p} = Sup_{0 \neq V \in H_0^{l,p'}} \frac{<S,V>_{-l,p}}{\|V\|_{l,p'}}$ where $< .,. >_{-l,p}$ signifies the duality pairing.
For the space of vector fields defined in Ω and on $\partial\Omega$, similar bold notations will be used and in the case of no confusion, we ignore the subscript Ω in all the above norms.
Define the symmetric square matrix $E(\mathbf{u})$, as

$$[E(\mathbf{u})]_{ij} = E_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}),$$

hence it is clear that $E(\mathbf{u}) = \frac{1}{2}(\ \nabla\mathbf{u} + (\nabla\mathbf{u})^T\ )$ and $tr(E(\mathbf{u})) = \nabla.\mathbf{u}$, where for a matrix $A_{n\times n} = [a_{ij}]_{n\times n}$, $tr(A) = \sum_{i=1}^n a_{ii}$.
These definitions are used to issue the following Lemma.

Lemma 3.1 *Given an open domain $\Omega \subseteq \Re^n$, $n \geq 2$, then there exists a positive constant C such that*

$$\|\boldsymbol{u}\|_1 \leq C|E(\boldsymbol{u})|_2, \qquad\qquad \forall \boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega) \equiv \boldsymbol{H}_0^{1,2}(\Omega).$$

Proof. As pointed out in [1], Korn demonstrated that $|\nabla\mathbf{u}|_2 \leq C_1|E(\mathbf{u})|_2$ for each $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$. On the other hand Poincare inequality states that

$$|\mathbf{u}|_2 \leq C_2|\nabla\mathbf{u}|_2,$$

consequently

$$\|\mathbf{u}\|_1^2 = |\mathbf{u}|_2^2 + |\nabla\mathbf{u}|_2^2 \leq (C_2^2 + 1)|\nabla\mathbf{u}|_2^2 \leq C_1^2(C_2^2 + 1)|E(\mathbf{u})|_2^2.$$

To complete the proof we choose $C = C_1\sqrt{C_2^2 + 1}$. $\diamondsuit$
If we note the cost functional $\mathcal{J}(\alpha, \mathbf{u}(\alpha)) = \|T^\alpha\|_{\odot,\Omega(\alpha)}$, $\gamma = \frac{\xi - \frac{2}{3}\mu}{2\mu}$ and assume that the technical condition $\xi \geq \frac{2}{3}\mu$, (see [11]), then there exists a positive constant C such that following inequality holds.

Lemma 3.2 $\|T^{\alpha}\|_{\odot,\Omega(\alpha)} \geq C\|\boldsymbol{u}\|_{1,\Omega(\alpha)}$.

Proof. Suppose $\mathbf{u} = (u_1, u_2)$, $\tau \equiv \frac{1}{2\mu}T^{\alpha} = E(\mathbf{u}) + \gamma\ tr(E(\mathbf{u}))\mathcal{I}$, where $\mathcal{I} = \mathcal{I}_2$, is the identity matrix with entries $\delta_{ij},\ i,j = 1,2$. The technical condition leads to $\gamma \geq 0$ and thus $2\sum_{i,j=1}^{2}(E_{ij}\ \delta_{ij}\ \gamma\ tr(E)\ ) \geq 0$, where $tr(E) = E_{11} + E_{22}$. Consider

$$\|\tau\|^2_{\odot,\Omega(\alpha)} = \int_{\Omega(\alpha)} \tau \odot \tau \ d\Omega = \int_{\Omega(\alpha)} \sum_{i,j=1}^{2} (E_{ij} + \delta_{ij}\ \gamma\ tr(E))^2 \ d\Omega$$

$$\geq \int_{\Omega(\alpha)} \sum_{i,j=1}^{2} (E_{ij})^2 \ d\Omega + \int_{\Omega(\alpha)} \sum_{i,j=1}^{2} (\delta_{ij}\ \gamma\ tr(E))^2 \ d\Omega$$

$$= \|E(\mathbf{u})\|^2_{\odot,\Omega(\alpha)} + \int_{\Omega(\alpha)} 4\gamma^2\ tr^2(E)\ d\Omega$$

$$\geq \|E(\mathbf{u})\|^2_{\odot,\Omega(\alpha)} = |E(\mathbf{u})|^2_{2,\Omega(\alpha)}.$$

Therefore $\|T^{\alpha}\|_{\odot,\Omega(\alpha)} \geq 2\mu|E(\mathbf{u})|_{2,\Omega(\alpha)}$ and by Lemma 3.1., the proof is completed.$\diamondsuit$

Theorem 3.3 *For a positive constant* C, $\mathcal{J}(\alpha, \boldsymbol{u}(\alpha)) \geq C\|\boldsymbol{u}\|_{1,\Omega(\alpha)}$.

Proof. The aim is achieved by the Lemma 3.2., when we put $\mathcal{J}(\alpha, \mathbf{u}(\alpha)) = \|T^{\alpha}\|_{\odot,\Omega(\alpha)}$.$\diamondsuit$

Up to here we introduce the weak variational formulation of (9) – (12) and it will be shown the existence and uniqueness of weak solution.

Let an operator $B : L^2 \longrightarrow L^2$ be defined by $p = BG$, where p is the solution of $\mathbf{U}.\nabla p = G$ in $\Omega(\alpha)$ and $p = 0$ on $\Gamma_{in}(\Omega(\alpha))$.

Take two bilinear forms $a(.,.) : \mathbf{H}_0^1 \times \mathbf{H}_0^1 \longrightarrow \Re$ and $b(.,.) : L^2 \times \mathbf{H}_0^1 \longrightarrow \Re$ defined as follows:

$$a(\mathbf{u}, \mathbf{v}) = \int_{\Omega(\alpha)} (\mu\nabla\mathbf{u} \odot \nabla\mathbf{v} + \nu\nabla.\mathbf{u}\nabla.\mathbf{v} + \varrho\mathbf{U}.\nabla\mathbf{u}.\mathbf{v})\ d\Omega,$$

$$b(q, \mathbf{v}) = \int_{\Omega(\alpha)} q\nabla.\mathbf{v}\ d\Omega.$$

One can verify by Green's Formula and above bilinear forms that (9) can be written as:

$$a(\mathbf{u}, \mathbf{v}) + b(p, \mathbf{v}) = <\mathbf{f}, \mathbf{v}>_{-1} \qquad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega(\alpha)). \tag{17}$$

Equation (10) consistently to the operator B yields

$$p = B(k^{-1}g) - B(k^{-1}\nabla.\mathbf{u}). \tag{18}$$

A function pair $(\mathbf{u}, p)$ is defined as the weak solution to the problem (9) – (12) when the function $\mathbf{u} \in \mathbf{H}_0^1(\Omega(\alpha))$ satisfies (17) and the function $p \in L^2(\Omega(\alpha))$ yields (18).

The existence of weak solution follows from the next Lemma in which assumed the first ingredient of $\mathbf{U}$ be strictly positive.

Lemma 3.4 *Let* $\gamma = \frac{1}{2}(|div(\varrho \boldsymbol{U})|_\infty + |div(k\boldsymbol{U})|_\infty)$ *and* $\mu - C\gamma > 0$ *for a constant* C. *Then there is a unique solution* $(\boldsymbol{u}, p) \in \boldsymbol{H}_0^1(\Omega(\alpha)) \times L^2(\Omega(\alpha))$ *of* (17), (18) *satisfying*

$$\|\boldsymbol{u}\|_1 + |p|_2 \leq C(\|\boldsymbol{f}\|_{-1} + |g|_2),$$

where $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega(\alpha))$ *and* $g \in L^2(\Omega(\alpha))$.

Proof. See [7].◇

The following Lemma verifies the boundedness of operator B.

Lemma 3.5 *Let* $G \in L^2(\Omega(\alpha))$. *Then there is a constant* C, *where* $p = BG$ *implies the inequality*

$$|BG|_2 \leq C|G|_2.$$

Proof. See [8].◇

To argue on the convergence of domains due to changing, we shall introduce a fixed domain $\hat{\Omega}$ such that $\bigcup_{\alpha \in \mathcal{A}_{ad}} \Omega(\alpha) \subseteq \hat{\Omega}$. It is adequate $\hat{\Omega}$ be the rectangular $[0, h_1] \times [0, h_2]$.

Suppose $\Omega_n = \Omega(\alpha_n)$, the convergence of Ω_n to $\Omega(\alpha)$ is defined in the following:

$$\Omega_n \longrightarrow \Omega \iff |\alpha - \alpha_n|_\infty = \max_{l_1 \leq x_1 \leq l_2} |\alpha(x_1) - \alpha_n(x_1)| \longrightarrow 0 \ \ ; \ n \to \infty.$$

Remark: It can be concluded from the Calderon's Extension Theorem in [9], for uniform Lipschitz domains $\{\Omega(\alpha), \alpha \in \mathcal{A}_{ad}\}$ in $\hat{\Omega}$ and positive integer s, there is a linear continuous extension operator $\mathbf{P} : \mathbf{H}^s(\Omega(\alpha)) \longrightarrow \mathbf{H}^s(\hat{\Omega})$ such that for each $\mathbf{f} \in \mathbf{H}^s(\Omega(\alpha))$,

$$\|\mathbf{Pf}\|_{s,\hat{\Omega}} \leq C\|\mathbf{f}\|_{s,\Omega(\alpha)}, \tag{19}$$

where the positive constant C depends only on the Lipschitz constant of the boundary of $\Omega(\alpha)$.

In the proof of our main theorem, Theorem 4.1., we will use the lower semi-continiuity property, where this property is represented as a useful result from [6] in the form of coming proposition.

Lemma 3.6 *Let* Ω *and* Ω_n *be bounded Lipschitz domain and* $(\boldsymbol{u}, \boldsymbol{u}_n) \in \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega_n)$ *such that* $\boldsymbol{u}_n \longrightarrow \boldsymbol{u}$. *Assume that* $F(t)$ *is continuous, convex, and nonnegative for* $t \in \Re$. *Then the following inequality yields:*

$$\int_\Omega F(\nabla \boldsymbol{u})\, d\Omega \leq \liminf_{n\to\infty} \int_{\Omega_n} F(\nabla \boldsymbol{u}_n)\, d\Omega.$$

Proof. See [6].◇

Note that $F : \Omega \subset X \longrightarrow \Re$ is (weakly) lower semicontinuous if for every sequence t_n in Ω, whenever $(t_n \rightharpoonup t)$ $t_n \longrightarrow t$ in X, then

$$\liminf_{n\to\infty} F(t_n) \geq F(t).$$

In the next section we will prove our principal issue by using all preliminaries presented in this and pervious sections.

4 Existence of optimal solution

Eventually we exhibit the main attainment of this paper in the following theorem. First assume that S_w be the set of all weak solutions $(\mathbf{u}(\alpha), p(\alpha)) \in \mathbf{H}_0^1(\Omega(\alpha)) \times L^2(\Omega(\alpha))$ satisfying (17), (18).

Theorem 4.1 *Let the admissible set of controls and states be defined as follows:*

$$\mathcal{S}_{ad} = \{(\alpha, \boldsymbol{u}(\alpha)) \in \mathcal{A}_{ad} \times \boldsymbol{H}_0^1(\Omega(\alpha)) : \mathcal{J}(\alpha, \boldsymbol{u}(\alpha)) < \infty \text{ and } (\boldsymbol{u}(\alpha), p(\alpha)) \in S_w\}.$$

Then the problem

$$\min_{(\alpha, \boldsymbol{u}(\alpha)) \in \mathcal{S}_{ad}} \mathcal{J}(\alpha, \boldsymbol{u}(\alpha))$$

has at least one optimal solution $(\alpha^*, \boldsymbol{u}(\alpha^*)) \in \mathcal{S}_{ad}$.

Proof. It is clear form Lemma 3.4, that $\mathcal{S}_{ad}$ is nonempty. We recall that $\Omega_n = \Omega(\alpha_n)$, $\mathbf{u}_n = \mathbf{u}(\alpha_n)$ and $p_n = p(\alpha_n)$. These definitions enable us to consider $\mathcal{J}(\alpha)$, for simplicity, instead of $\mathcal{J}(\alpha, \mathbf{u}(\alpha))$. Due to

$$0 < \inf_{(\alpha, \mathbf{u}(\alpha)) \in \mathcal{S}_{ad}} \mathcal{J}(\alpha) < \infty,$$

therefore a minimizing sequence $\{(\alpha_n, \mathbf{u}_n)\}$ in $\mathcal{S}_{ad}$, exists such that

$$\lim_{n \to \infty} \mathcal{J}(\alpha_n) = \inf_{(\alpha, \mathbf{u}(\alpha)) \in \mathcal{S}_{ad}} \mathcal{J}(\alpha).$$

Using Arzela-Ascoli Theorem [4] and the characteristic of $\mathcal{A}_{ad}$, arise that there is a subsequence of $\{\alpha_n\}$, say again $\{\alpha_n\}$, converges uniformly in $[l_1, l_2]$ to an $\alpha^* \in \mathcal{A}_{ad}$, means that $\alpha_n \longrightarrow \alpha^*$ as n tends to infinity.
Uniformly boundedness of $\mathbf{u}_n$ in $\mathbf{H}_0^1(\Omega_n)$ is deduced from Theorem 3.3, and $\mathcal{J}(\alpha) < \infty$, additionally, the uniform extension property empowers us to take an extension $\hat{\mathbf{u}}_n$ of $\mathbf{u}_n$ to $\hat{\Omega}$ in such way that

$$\|\hat{\mathbf{u}}_n\|_{1,\hat{\Omega}} \leq C \|\mathbf{u}_n\|_{1,\Omega_n},$$

where C is a positive constant independent of n.
Consequently $\|\hat{\mathbf{u}}_n\|_{1,\hat{\Omega}}$ is uniformly bounded in $\mathbf{H}_0^1(\hat{\Omega})$. The latter property holds on $|\hat{p}_n|_{2,\hat{\Omega}}$, which is resulted from uniformly boundedness of $\|\hat{\mathbf{u}}_n\|_{1,\hat{\Omega}}$ and Lemma 3.4.
So we can derive from the sequence $\{(\hat{\mathbf{u}}_n, \hat{p}_n)\}$ a subsequence, say again $\{(\hat{\mathbf{u}}_n, \hat{p}_n)\}$, converges to $(\hat{\mathbf{u}}, \hat{p})$ in $\mathbf{H}^1(\hat{\Omega}) \times L^2(\hat{\Omega})$, marked as follows:

$$\hat{\mathbf{u}}_n \rightharpoonup \hat{\mathbf{u}} \qquad \textit{in} \;\; \mathbf{H}^1(\hat{\Omega}) \tag{20}$$

$$\hat{\mathbf{u}}_n \to \hat{\mathbf{u}} \qquad \textit{in} \;\; \mathbf{L}^2(\hat{\Omega}) \tag{21}$$

$$\hat{p}_n \rightharpoonup \hat{p} \qquad\qquad in \;\; L^2(\hat{\Omega}). \tag{22}$$

Let $\mathbf{u}^* = \hat{\mathbf{u}}|_{\Omega(\alpha^*)}$ and $p^* = \hat{p}|_{\Omega(\alpha^*)}$. If we are able to demonstrate that $(\alpha^*, \mathbf{u}^*) \in \mathcal{S}_{ad}$, then on the other word we have shown that $(\mathbf{u}^*, p^*)$ is a solution of (17), (18), in fact we have proved $\mathcal{S}_{ad}$ is weakly closed.
Defining the two following function spaces pursues our intention. Let $U(\eta)$ be a neighborhood of an arbitrary set η, and $\eta_n = \Gamma_{low} \bigcup \Gamma_{wing}(\alpha_n)$ and $\eta_* = \Gamma_{low} \bigcup \Gamma_{wing}(\alpha^*)$.
Let $S_n = \{\mathbf{w} \in \mathbf{C}^\infty(\bar{\Omega}_n); \mathbf{w}|_{U(\eta_n)} = 0\}$ and $S_* = \{\mathbf{w} \in \mathbf{C}^\infty(\bar{\Omega}(\alpha^*)); \mathbf{w}|_{U(\eta_*)} = 0\}$.
One can verifies that $\mathbf{H}_0^1(\Omega_n)$ is the closure of S_n in $\mathbf{H}^1(\Omega_n)$, and $\mathbf{H}_0^1(\Omega(\alpha^*))$ is the closure of S_* in $\mathbf{H}^1(\Omega(\alpha^*))$.
We extend each element of $\mathbf{H}_0^1(\Omega(\alpha^*))$ by zero in $\bar{\hat{\Omega}} - \Omega(\alpha^*)$.
Take $\mathbf{w} \in S_*$ and replace $\mathbf{w}$ instead of $\mathbf{v}$ in (17) on Ω_n, then

$$a(\mathbf{u}_n, \mathbf{w}) + b(p_n, \mathbf{w}) =< \mathbf{f}, \mathbf{w} >_{-1} \qquad\qquad \mathbf{w} \in S_*.$$

Now we investigate the bilinear forms separately.

$$a(\mathbf{u}_n, \mathbf{w}) = \int_{\Omega_n} (\mu \nabla \mathbf{u}_n \odot \nabla \mathbf{w} + \nu \nabla . \mathbf{u}_n \nabla . \mathbf{w} + \varrho \mathbf{U} . \nabla \mathbf{u}_n . \mathbf{w}) \; d\Omega,$$

Consider the first term of $a(.,.)$

$$\int_{\Omega_n} \mu \nabla \mathbf{u}_n \odot \nabla \mathbf{w} \; d\Omega := (\; extend \;\; \mathbf{u}_n \; to \;\; \hat{\Omega}) =$$

$$\int_{\hat{\Omega}} \mu \nabla \hat{\mathbf{u}}_n \odot \nabla \mathbf{w} \; d\Omega := (\; consider \;\; (19) \; and \; let \;\; n \to \infty) =$$

$$\int_{\hat{\Omega}} \mu \nabla \hat{\mathbf{u}} \odot \nabla \mathbf{w} \; d\Omega := (\; restrict \;\; \mathbf{w} \; on \;\; \Omega(\alpha^*)) =$$

$$\int_{\Omega(\alpha^*)} \mu \nabla \hat{\mathbf{u}}|_{\Omega(\alpha^*)} \odot \nabla \mathbf{w} \; d\Omega := (\; use \; the \;\; definition \; of \;\; \mathbf{u}^*) =$$

$$\int_{\Omega(\alpha^*)} \mu \nabla \mathbf{u}^* \odot \nabla \mathbf{w} \; d\Omega,$$

other terms of $a(.,.)$ are evaluated in similar manner, therefore

$$a(\mathbf{u}_n, \mathbf{w}) \longrightarrow a(\mathbf{u}^*, \mathbf{w}) \; ; \qquad\qquad n \to \infty,$$

and by the same way, one can show

$$b(p_n, \mathbf{w}) \longrightarrow b(p^*, \mathbf{w}) \; ; \qquad\qquad n \to \infty.$$

These attainments state that $\mathbf{u}^*$ is a solution of

$$a(\mathbf{u}^*, \mathbf{w}) + b(p^*, \mathbf{w}) =< \mathbf{f}, \mathbf{w} >_{-1} \qquad\qquad \forall \mathbf{w} \in S_* \; . \tag{23}$$

To show that p^* satisfies the following relation

$$p^* = B(k^{-1}g) - B(k^{-1}\nabla.\mathbf{u}^*), \tag{24}$$

we perform the following process.
First we define

$$\mathcal{K}(\Omega_n, \lambda) = \int_{\Omega_n} \mid [p_n - (\ B(k^{-1}g) - B(k^{-1}\nabla.\mathbf{u}_n)\ )]\lambda \mid d\Omega,$$

where $\lambda \in \Lambda = \{\lambda \in L^2(\hat{\Omega}); \lambda = 0 \ \ on\ \bar{\hat{\Omega}} - \Omega(\alpha^*)\}$.
Since $(\mathbf{u}_n, p_n)$ is a weak solution of (17), (18), hence $(\mathbf{u}_n, p_n)$ satisfies (18) and it is clear that $\mathcal{K}(\Omega_n, \lambda)$ is zero.
Now we recall $\mathcal{K}(\Omega_n, \lambda)$,

$$\int_{\Omega_n} \mid [p_n - (\ B(k^{-1}g) - B(k^{-1}\nabla.\mathbf{u}_n)\ )]\lambda \mid d\Omega := (\ extend \ \ \mathbf{u}_n \ \ and\ p_n \ \ to \ \ \hat{\Omega}) =$$

$$\int_{\hat{\Omega}} \mid [\hat{p}_n - (\ B(k^{-1}g) - B(k^{-1}\nabla.\hat{\mathbf{u}}_n)\ )]\lambda \mid d\Omega$$

$$:= (\ consider\ (19) - (21)\ , continuity\ of\ B\ by\ Lemma\ 3.6,\ and\ let\ n \to \infty) =$$

$$\int_{\hat{\Omega}} \mid [\hat{p} - (\ B(k^{-1}g) - B(k^{-1}\nabla.\hat{\mathbf{u}})\ )]\lambda \mid d\Omega$$

$$:= (\ restrict\ \lambda \ \ on \ \ \Omega(\alpha^*) \ \ and\ use\ the \ \ definition\ of \ \ \mathbf{u}^*\ and \ \ p^*) =$$

$$\int_{\Omega(\alpha^*)} \mid [p^* - (\ B(k^{-1}g) - B(k^{-1}\nabla.\mathbf{u}^*)\ )]\lambda \mid d\Omega.$$

Therefore

$$\mathcal{K}(\Omega_n, \lambda) \longrightarrow \int_{\Omega(\alpha^*)} \mid [p^* - (\ B(k^{-1}g) - B(k^{-1}\nabla.\mathbf{u}^*)\ )]\lambda \mid d\Omega\ ; \qquad n \to \infty,$$

this issue expresses that

$$[p^* - (\ B(k^{-1}g) - B(k^{-1}\nabla.\mathbf{u}^*)\ )]\lambda = 0 \quad \forall \lambda \in \Lambda,$$

consequently (23) is satisfied.
Equations (22) and (23) succeed that $(\alpha^*, \mathbf{u}^*)$ is in $\mathcal{S}_{ad}$ and thus $\mathcal{S}_{ad}$ is weakly closed.
Following Lemma 3.6, $\mathcal{J}(\alpha)$ is weakly lower semicontinuous and the proof of the theorem is ended by the fact that

$$\mathcal{J}(\alpha^*) \leq \liminf_{n \to \infty} \mathcal{J}(\alpha_n) = \lim_{n \to \infty} \mathcal{J}(\alpha_n) = \inf_{(\alpha, \mathbf{u}(\alpha)) \in \mathcal{S}_{ad}} \mathcal{J}(\alpha),$$

this implies $\mathcal{J}(\alpha^*) = \inf_{(\alpha, \mathbf{u}(\alpha)) \in \mathcal{S}_{ad}} \mathcal{J}(\alpha)$, it means that $(\alpha^*, \mathbf{u}^*)$ is an optimal solution.$\diamondsuit$

References

[1] R. G. Duran and M. A. Muschietti, The Korn Inequality for Jones Domains, *Electronic J. of Differential Equations,* **127**, (2004), 1-10.

[2] B. Farhadinia and M. H. Farahi, Optimal Shape Design of an Almost Straight Nozzle, *International J. of Applied Mathematics,* **3**, (2005), 319-333.

[3] B. Farhadinia, M. H. Farahi and J. A. Esfahani, Optimal Shape Design of a Nozzle Governed by Navier-Stokes Equations, *Submitted.*

[4] G. B. Folland, *Real Analysis*, John Wiley and Sons, New York, (1985).

[5] M. Gunzburger and H. Kim, Existence of an Optimal Solution of a Shape Control Problem for the Stationary Navier-Stokes Equations ,*SIAM J. Control Optim.,* **3** , (1998), 895-909.

[6] A. D. Ioffe, On Lower Semicontinuity of Integral Functionals. I,*SIAM. J. Control Optim.,* **4**, (1977), 521-538.

[7] J. R. Kweon and R. B. Kellogg, Compressible Navier-Stokes Equations in a Bounded Domain with Inflow Boundary Condition, *SIAM. J. Math. Anal.,* **1**, (1997), 94-108.

[8] J. R. Kweon and R. B. Kellogg, Regularity of Solutions to the Navier-Stokes Equations for Compressible Barotropic Flows on a Polygon, *Arch. Rational Mech. Anal.,* **163**, (2002), 35-64.

[9] J. Marti, *Introduction to Sobolev Spaces and Finite Element Solution of Elliptic Boundary Value Problems,* Academic Press, London, (1986).

[10] B. Mohammadi, *Applied Shape Optimization for Fluids,* Clarendon Press, Oxford, (2001).

[11] K. Pileckas and W. M. Zajaczkowski, On the Free Boundary Problem for Stationary Compressible Navier-Stokes Equations, *Commun. Math. Phys.,* **129**, (1990), 169-204.

[12] O. Pironneau, *Optimal Shape Design for Elliptic Systems,* Springer-Verlag, New York, Berlin, Heidelberg,Tokyo, (1984).

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