Int. J. Contemp. Math. Sci., Vol. 2, 2007, no. 14, 685 - 691

Growth Sequence of Free Product of Alternating Groups

A. Erfanian[1]

Department of Mathematics
Ferdowsi University of Mashhad
Mashhad, Iran
erfanian@math.um.ac.ir

Abstract

For a finitely generated group G, we denote G^n as the direct product of n copies of G. The growth sequence of G is the sequence $\{d(G^n)\}_{n\geq 1}$, where $d(G^n)$ is the minimum number of generators of G^n. In this paper, we investigate the growth sequence of G, when G is the free product of alternating groups. In fact, we prove that

$$d\left((A_n * A_m)^{h(2,A_n)h(k,A_m)}\right) \leq k+2,$$

for all n, $m \geq 5$ and $k \geq 2$, where $h(2, A_n)$ is the maximum number t such that $d(A_n^t) = 2$ and similarly, $h(k, A_m)$ is the maximum number s such that $d(A_m^s) = k$. Moreover, we will consider the case $k = 2$ and prove that $d((A_n * A_m)^t) = 4$, for all $1 \leq t \leq h(2, A_n)h(2, A_m)$ and $n, m \geq 5$. We have also confirmed the above results by several examples in find section.

Mathematics Subject Classification: Primary: 20F05, 20E32; Secondary: 20D08

Keywords: Minimum number of generators, growth sequences, free product, alternating groups

1 Introduction

Let G be a finitely generated group, and G^n be the n-th direct power of G. The growth sequence of G is the sequence $\{d(G^n)\}_{n\geq 1}$, where $d(G^n)$ is

[1]The author thank the *Centre of Excellency in Analysis on Algebraic Structures*, Ferdowsi University of Mashhad for supporting the part grant on this work.

the minimum number of generators of G^n. There are some known results on the growth sequence of finite groups in a series of papers by J.Wiegold in [4,7,9,10]. He proved that if G is a perfect group (i.e. $G = G'$), then the growth sequence of G increases roughly logorithmically in n and if it is not perfect then $d(G^n) = nd(\frac{G}{G'})$, for large enough n. D.Meier and J.Wiegold [4] have also shown that if G is a non-abelian simple group of order s with automorphism group of order a, then for sufficiently large n

$$log_s n + log_s a - 1 + \theta(n) \leq d(G^n) \leq log_s n + log_s a + 1 + \psi(n),$$

where θ and ψ are sequences tending to 0 as $n \to \infty$. So, for large n, $d(G^n)$ must take one of at most three values near $log_s n + log_s a$.
In 1995, J.Wiegold and the author [2] gave a formula for the exact values $d(G^n)$, in terms of the Aut(G)- classes, for every finite non-abelian simple group G and every $n \geq 1$.

For infinite groups, the situation is less clear. If G is perfect, the growth sequences of G is bounded above by a logarithmic function of n and if G is not perfect, then again by [11] we have

$$d(G^n) = nd(\frac{G}{G'}),$$

for large enough n. In 1978, J. Wiegold and J. S. Wilson [11] announced some upper bounds for the growth sequences of finitely generated perfect and imperfect groups. Later on, A. G. R. Stewart and J. Wiegold (1989) in [5] using a different technique found a better upper bound for the growth sequences of finitely generated perfect groups.

One of the interesting cases of such groups is a free product of two finite non-abelian simple groups. This article will provide some precise values and upper bounds for the growth sequences of the free product $A_n * A_m$, for every n and $m \geq 5$.

2 Some Definitions and Basic Results

In this section we give some definitions and known results, which are necessary for our purposes. We start with the following two concepts.

Definition 2.1 We denote $\phi_n(G)$ to be the total number of distinct n-tuples of elements of G which generate G, and call it the n^{th} *Eulerian function*. If G can not be generated by n elements, then $\phi_n(G) = 0$

Definition 2.2 For any fixed element x of a group G, the size of the set

$$\{y \in G \mid < x, y >= G\}$$

is denoted by $cs(x)$, and called the *cospread* of x in G. Clearly $cs(x) = 0$ if G can not be generated by elements x and y, for every $y \in G$.

P. Hall [3] observed that for a non-abelian simple group G, and $k \geq d(G)$

$$\begin{aligned} h(n, G) &= \max\{k \ : \ d(G^k) = n\} \\ &= \frac{1}{|Aut(G)|} \sum_{H \leq G} \mu(H)|H|^n, \end{aligned}$$

where μ is the Möbius function of the subgroup lattice of G. It is also shown that

$$h(n, G) = \frac{\phi_n(G)}{|AutG|},$$

for every non-abelian finite simple group G. One may check that

$$h(2, G) = \frac{1}{|Aut(G)|} \sum_{x \in G} cs(x).$$

The following result from [6], gives a lower bound for the Eulerian function of G.

Theorem 2.3 (J. Thevenaz [6]) Let G be a non-trivial finite simple group and n be a positive integer. Then

$$\phi_n(G) \geq |G|^n - \sum_i |M_i|^n,$$

where the sum is taken over all maximal subgroups M_i of G.

The following lemmas follow from the definition of $\phi_n(G)$, Theorem 2.3 and the formula given by P. Hall which were stated above. The proofs are omitted.

Lemma 2.4 For any finite non-abelian simple group G,

$$h(m, G) > \frac{|G|^m - \sum_i |M_i|^m}{|Aut(G)|}$$

Lemma 2.5 If G is any finite group and $m \geq 2$, then

$$\phi_{m+1}(G) > |G|\phi_m(G).$$

Now, we are in a position to state our main results as follows.

Theorem A. Let $G = A_n * A_m$ be the free product of alternating groups A_n and A_m, for $n, m \geq 5$. Then $d(G^t) = 4$, for all integers t such that $1 \leq t \leq h(2, A_n)h(2, A_m)$.

Theorem B. Let $G = A_n * A_m$, where n and $m \geq 5$. Then

$$d\left((A_n * A_m)^{h(2,A_n)h(k,A_m)}\right) \leq k + 2,$$

for all $k \geq 2$.

3 Proof of Theorem A

To prove Theorem A, we need the following simple lemma.

Lemma 3.1. There exists an element x in A_n such that $cs(x) \geq h(2, A_n)$, for all $n \geq 5$.

Proof. Assume that for all x in A_n, $cs(x) < h(2, A_n)$. Then we have

$$\begin{aligned} h(2, A_n) &= \frac{1}{|Aut\ A_n|} \sum_{x \in A_n} cs(x) < \frac{1}{|Aut\ A_n|} \sum_{x \in A_n} h(2, A_n) \\ &= \frac{|A_n|}{|Aut\ A_n|} h(2, A_n) = \begin{cases} \frac{1}{2} h(2, A_n) & \text{if } n \neq 6 \\ \frac{1}{4} h(2, A_n) & \text{if } n = 6 \end{cases} \end{aligned}$$

which is a contradiction.

Now, we are able to prove Theorem A.

Proof of Theorem A. Suppose that $t = h(2, A_n)$ and $s = h(2, A_m)$. Then $d(A_n^t) = 2$ and $d(A_m^s) = 2$. So, there are elements $a_1, a_2, \ldots, a_t$ and $b_1, b_2, \ldots, b_t$ such that $A_n =< a_i, b_i >$, for all $1 \leq i \leq t$, and $n \geq 5$. Moreover, there is no automorphism of A_n which transfer (a_i, b_i) to (a_j, b_j) for all i and j with $1 \leq i \neq j \leq t$. Now, by Lemma 3.2, there exists an element x in A_m such that $cs(x) \geq s$. Thus, there are distinct elements $d_1, d_2, \ldots, d_s$ in A_m such that $A_m =< x, d_i >$, for all $1 \leq i \leq s$. Therefore, we can define the following elements of $A_n^{h(2,A_n)h(2,A_m)}$.

$$\begin{array}{lcl} u &=& (a_1, a_2, \ldots, a_t, a_1, a_2, \ldots, a_t, \ldots, a_1, a_2, \ldots, a_t) \\ v &=& (b_1, b_2, \ldots, b_t, b_1, b_2, \ldots, b_t, \ldots, b_1, b_2, \ldots, b_t) \\ w &=& (x, x, \ldots, x, x, x, \ldots, x, \ldots, x, x, \ldots, x) \\ z &=& (d_1, d_1, \ldots, d_1, d_2, d_2, \ldots, d_2, \ldots, d_s, d_s, \ldots, d_s) \end{array}$$

It is clear that each column generate $A_n * A_m$, and so $(A_n * A_m)^{ts}$ can be generated by the above four elements. Thus

$$d((A_n * A_m)^{ts}) \leq 4.$$

But, we know that $d(A_n * A_m) = 4$, and hence

$$d((A_n * A_m)^{ts}) = 4.$$

Thus, the proof of Theorem A is completed.

Proposition 3.2. For large enough n and m,

$$d\left((A_n * A_m)^{\frac{n!m!}{25}}\right) = 4.$$

Proof. By [1], we have $h(2, A_n) > \frac{n!}{5}$, for large n. Thus, the proof follows from Theorem A directly.

4 Proof of Theorem B

Using similar technique, as in the proof of Theorem A, we may give $k+2$ generators for $G^{h(2,A_n)h(k,A_m)}$, where $G = A_n * A_m$, $n, m \geq 5$, and $k \geq 2$. Suppose that $t = h(2, A_n)$ and $s = h(k, A_m)$, then $d(A_n^t) = 2$ and $d(A_m^s) = k$. If $k = 2$, then the proof is clear by Theorem A , and so $d(G^{ts}) = k+2$. Now, assume that $k > 2$. Then, we consider the following k generators for A_m^s:

$$x_1 = (a_{11}, a_{12}, \ldots, a_{1s})$$

$$x_2 = (a_{21}, a_{22}, \ldots, a_{2s})$$

$$\vdots$$

$$x_k = (a_{k1}, a_{k2}, \ldots, a_{ks}).$$

By Lemma 3.1, there exists an element x in A_n such that $cs(x) \geq h(2, A_n)$. Thus, we have $< x, d_i >= A_n$ for all $1 \leq i \leq h(2, A_n)$. Now, we can easily see that the following $k+2$ elements generate G^{ts}.

$$\begin{array}{lcl}
y_1 & = & (a_{11}, \ldots, a_{1s}, a_{11}, \ldots, a_{1s}, \ldots, a_{11}, \ldots, a_{1s}) \\
y_2 & = & (a_{21}, \ldots, a_{2s}, a_{21}, \ldots, a_{2s}, \ldots, a_{21}, \ldots, a_{2s}) \\
\vdots & & \\
y_k & = & (a_{k1}, \ldots, a_{ks}, a_{k1}, \ldots, a_{ks}, \ldots, a_{k1}, \ldots, a_{ks}) \\
y_{k+1} & = & (x, \ldots, x, x, \ldots, x, \ldots, x, \ldots, x) \\
y_{k+2} & = & (d_1, \ldots, d_1, d_2, \ldots, d_2, \ldots, d_t, \ldots, d_t).
\end{array}$$

Because, each column generates $A_n * A_m$ and there is no automorphism of A_n or A_m transferring one column to another. So it follows that $d(G^{ts}) \leq k+2$, which completes the proof.

Finally, we give some examples and state a conjecture as the following.

Examples

(i) Assume that $G = A_5 * A_5$, then we have $h(2, A_5) = 19$, $h(3, A_5) = 1,688$, $h(4, A_5) = 109,549$, $h(5, A_5) = 6,461,040$ by [1]. Thus, using Theorems A and B, we must have the following evaluations:

$$d(G^{361}) = 4 \quad , \quad 4 \leq d(G^{31,692}) \leq 5$$
$$d(G^{2,081,431}) \leq 6 \quad , \quad d(G^{122,816,760}) \leq 7$$

(ii) For $G = A_5 * A_6$, we have $h(2, A_6) = 53$, $h(3, A_6) = 30,132$, $h(4, A_6) = 11,538,875$ and $h(5, A_6) = 4,191,989,400$. So, we can see that

$$d(G^{1,007}) = 4 \quad , \quad 4 \leq d(G^{572,508}) \leq 5$$
$$d(G^{219,238,625}) \leq 6 \quad , \quad d(G^{79,647,798,600}) \leq 7$$

(iii) Suppose that $G = A_7 * A_7$, then we have $h(2, A_7) = 916$, $h(3, A_7) = 3,077,056$, $h(4, A_7) = 7,972,539,694$. Thus, we have

$$d(G^{839,056}) = 4 \quad , \quad 4 \leq d(G^{2,818,583,296}) \leq 5$$
$$d(G^{7,302,846,359,704}) \leq 6 \quad ,$$

Using GAP, we have a strong evidence that the following conjecture is true.

Conjecture. Let $G = A_n * A_n$ be the free squared product of A_n, $n \geq 5$. Then

$$d(G^{|A_n|^k}) = k+3,$$

for all $k \geq 1$.

If $k = 1$ and n is large, then we have $h(2, A_n) > \frac{n!}{5}$, by [1]. So

$$h(2, A_n)^2 > (\frac{n!}{5})^2 > |A_n|,$$

and the conjecture is true in this case. Moreover, using GAP program and spending a lot of time for running it, I have been able to confirm the conjecture for $k = 1$ and $5 \leq n \leq 10$.

5 References

[1] Erfanian, A., A note on growth sequences of alternating groups, Arch. Math. **78**, 257-262 (2002).

[2]Erfanian, A. and Wiegold, J., A note on growth sequence of finite simple groups, Bull. Austral. Math. Soc. Vol. **51** , 495-499 (1995).

[3] Hall, P., The Eulerian function of group, Quart. J. Math. (Oxford) **7**, 134-151 (1936).

[4] Meier, D. and Wiegold, J., Growth sequence of finite groups V, J. Austral. Math. Soc. Ser A **31**, 374-375 (1981).

[5] Stewart, A. G. R. and Wiegold, J., Growth sequence of finite groups II, Bull. Austral. Math. Soc. Vol. **40**, 323-329 (1989)

[6] Thevenaz, J., Maximal subgroups of direct products, Journal of Algebra **198**, 352-361 (1997).

[7] Wiegold, J., Growth sequence of finite groups, J. Austral. Math. Soc. **17**, 133-141 (1974).

[8] Wiegold, J., Growth sequence of finite groups II, J. Austral. Math. Soc. **20**, 225-229 (1975).

[9] Wiegold, J., Growth sequence of finite groups III, J. Austral. Math. Soc. **25**, 142-144 (1978).

[10] Wiegold, J., Growth sequence of finite groups IV, J. Austral. Math. Soc. **29**, 14-16 (1980).

[11] Wiegold, J. and Wilson, J. S., Growth sequence of finitely generated groups, Arch. Math. **30**, 337-343 (1978).

Received: November 8, 2006