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On Subclass of Close-to-Convex Functions

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Abstract

In this paper the authors studied the coefficient estimate of a class of functions starlike with respect to k-symmetric points defined by derivative operators D_λ^n introduced by Al-Shaqsi and Darus [6]. The integral representation and several coefficient inequalities of functions belonging to this class are obtained.

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1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m, \tag{1.1}$$

which are analytic in the unit disk $\mathbf{U} = \{z : |z| < 1\}$.

Also let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all functions which are univalent in $\mathbf{U}$. A function $f \in \mathcal{A}$ is said to be starlike, denoted by S^* if $\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0$.

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The authors introduced the following differential operator (see [6]):

$$D^0_\lambda f(z) = (1-\lambda)f(z) + \lambda z f'(z) = D_\lambda f(z), \quad \lambda \geq 0, \tag{1.2}$$
$$D^1_\lambda f(z) = (1-\lambda)zf'(z) + \lambda z(zf'(z))', \tag{1.3}$$
$$D^n_\lambda f(z) = D_\lambda \left(\frac{z(z^{n-1}f(z))^n}{n!} \right), \ (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{1.4}$$

If the function f is given by (1.1), then we write

$$D^n_\lambda f(z) = z + \sum_{m=2}^{\infty} \big[1 + \lambda(m-1)\big] C(n,m) a_m z^m, \tag{1.5}$$

where

$$C(n,m) = \binom{m+n-1}{n} = \frac{\prod_{j=1}^{m-1}(j+n)}{(m-1)!}, \qquad m \geq 2. \tag{1.6}$$

Sakaguchi [7] once introduced a class S^*_S of functions starlike with respect to symmetric points, which consists of functions $f \in \mathcal{S}$ satisfying the inequality

$$\mathrm{Re}\left\{ \frac{zf'(z)}{f(z) - f(-z)} \right\} > 0, \qquad z \in \mathbf{U}. \tag{1.7}$$

Several different authors had studied the class of Sakaguchi [7] and discussed this class and its subclasses (see[1,3-5, 8-13]). Chand and Singh [9] for example, had introduced a class $S^{*(k)}_S$ of functions starlike with respect to k-symmetric points, which consists of functions $f \in \mathcal{S}$ satisfying the inequality

$$\mathrm{Re}\left\{ \frac{zf'(z)}{f_k(z)} \right\} > 0, \qquad z \in \mathbf{U}, \tag{1.8}$$

where

$$f_k(z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{-\nu} f(\varepsilon^\nu z), \qquad (k \geq 1; \varepsilon^k = 1). \tag{1.9}$$

In [6] the authors studied the class $K^{(k)}_S(n, \lambda; \phi(z))$ consisting of functions $f \in \mathcal{A}$ for which

$$\frac{z(D^n_\lambda f(z))'}{D^n_\lambda f_k(z)} \prec \phi(z), \qquad z \in \mathbf{U}, \tag{1.10}$$

where $D_\lambda^n f(z)$ given by (1.4), and

$$D_\lambda^n f_k(z) \;=\; D_\lambda\left(\frac{1}{k}\sum_{\nu=0}^{k-1}\varepsilon^{-\nu}\frac{z(z^{n-1}f(\varepsilon^\nu z))^n}{n!}\right) \tag{1.11}$$

where $k \geq 1$, $\lambda \geq 0$ and $n \in \mathbb{N}_0$.

In this paper we define the class $K_S^{(k)}(n, \lambda, \alpha, \beta)$ consisting of analytic functions as the following:

Definition 1.1 *Let $f \in \mathcal{A}$ and satisfies the following inequality:*

$$\left|\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f_k(z)} - 1\right| < \beta\left|\frac{\alpha z(D_\lambda^n f(z))'}{D_\lambda^n f_k(z)} + 1\right|, \qquad z \in \boldsymbol{U}, \tag{1.12}$$

where $0 \leq \alpha \leq 1, 0 < \beta \leq 1$, $k \geq 1$, $\lambda \geq 0$ and $n \in \mathbb{N}_0$.

We note that $K_S^{(2)}(0,0,1,1) \equiv S_S^*$ [7], $K_S^{(2)}(0,0,\alpha,\beta) \equiv S_S^{(2)}(\alpha,\beta)$ [11] and $K_S^{(k)}(0,0,\alpha,\beta) \equiv S_S^{(k)}(\alpha,\beta)$ [1].

In fact, the class $K_S^{(k)}(n, \lambda, \alpha, \beta)$ is a special case of the class $K_S^{(k)}(n, \lambda; \phi)$ studied in [6] when $\phi(z) = (1+\beta z)/(1-\alpha\beta z)$.

2 Coefficient Estimates

First, we need a lemma of Lakshminarasimhan [10].

Lemma 2.1 *Let $H(z)$ be analytic in $\boldsymbol{U}$ and satisfy the condition*

$$\left|\frac{1-H(z)}{1+\alpha H(z)}\right| < \beta, \qquad z \in \boldsymbol{U}, \tag{2.1}$$

where $0 \leq \alpha \leq 1, 0 < \beta \leq 1$, with $H(0) = 1$. Then we have

$$H(z) = \frac{1 - z\phi(z)}{1 + \alpha z\phi(z)}, \tag{2.2}$$

where $\phi(z)$ is analytic in $\boldsymbol{U}$ and $|\phi(z)| \leq \beta$ for $z \in \boldsymbol{U}$. Conversely any function $H(z)$ given by (2.2) above is analytic in $\boldsymbol{U}$ and satisfies (2.1) .

Next we give the following lemma, which shall be used to obtain the coefficient estimates for functions in the class $K_S^{(k)}(n, \lambda, \alpha, \beta)$.

Lemma 2.2 *Let f and g belong to $\mathcal{A}$ and satisfy*

$$\left|\frac{z(D_\lambda^n f(z))'}{D_\lambda^n g(z)} - 1\right| < \beta\left|\frac{\alpha z(D_\lambda^n f(z))'}{D_\lambda^n g(z)} + 1\right|,$$

where $0 \leq \alpha \leq 1, 0 < \beta \leq 1$, $k \geq 1$, $\lambda \geq 0$ and $n \in \mathbb{N}_0$, with f given by (1.1) and $g(z) = z + \sum_{m=2}^{\infty} b_m z^m$.

Then for $m \geq 2$,

$$\begin{aligned} &\Big[\big[1+\lambda(m-1)\big]C(n,m)(|ma_m - b_m|)\Big]^2 \\ &\leq \quad 2(\alpha\beta^2+1)\sum_{j=2}^{m-1} j\big[1+\lambda(j-1)\big]C(n,j)|a_j||b_j| \qquad (|a_1| = |b_1| = 1). \end{aligned} \tag{2.3}$$

Proof. We use the same method of Sudharsan et al [11]. By Lemma 2.1 we have

$$\frac{z(D_\lambda^n f(z))'}{D_\lambda^n g(z)} = \frac{1 - z\phi(z)}{1 + \alpha z\phi(z)} \tag{2.4}$$

where $\lambda \geq 0$, $n \in \mathbb{N}_0$, $\phi(z)$ is analytic in $\mathbf{U}$ and $|\phi(z)| \leq \beta$ for $z \in \mathbf{U}$.
Then

$$\big[\alpha z(D_\lambda^n f(z))' + D_\lambda^n g(z)\big] z\phi(z) = D_\lambda^n g(z) - z(D_\lambda^n f(z))'.$$

Now if

$$\psi(z) = z\phi(z) = \sum_{m=1}^{\infty} t_m z^m,$$

then

$$|\psi(z)| \leq \beta|z| \qquad\qquad z \in \mathbf{U}.$$

Therefore

$$\begin{aligned} &\left[(\alpha+1)z + \sum_{m=2}^{\infty}\big[1+\lambda(m-1)\big]C(n,m)\Big[\alpha m a_m + b_m\Big]z^m\right]\left[\sum_{m=1}^{\infty} t_m z^m\right] \\ &= \sum_{m=2}^{\infty}\big[1+\lambda(m-1)\big]C(n,m)\Big[b_m - m a_m\Big]z^m. \end{aligned} \tag{2.5}$$

Equating the coefficients of z^m in (2.5), we have

$$\begin{aligned}&\left[1+\lambda(m-1)\right]\left[b_m - ma_m\right]C(n,m)\\&=(\alpha+1)t_{m-1}+\left[1+\lambda\right]\left[\alpha 2a_2+b_2\right]C(n,2)t_{m-2}+\cdots\\&+\left[1+\lambda(m-2)\right]\left[\alpha(m-1)a_{m-1}+b_{m-1}\right]C(n,m-1)t_1.\end{aligned}$$

Thus the coefficient combination on the right side of (2.5) depends only upon the coefficients combination

$$\left[1+\lambda\right]\left[\alpha 2a_2+b_2\right]C(n,2)+\cdots+\left[1+\lambda(m-2)\right]\left[\alpha(m-1)a_{m-1}+b_{m-1}\right]C(n,m-1)$$

on the left side.

Hence for $m \geq 2$ we can write

$$\begin{aligned}&\left[(\alpha+1)z+\sum_{j=2}^{m-1}\left[1+\lambda(j-1)\right]C(n,j)\left[\alpha ja_j+b_j\right]z^j\right]\psi(z)\\&=\sum_{j=2}^{m}\left[1+\lambda(j-1)\right]C(n,j)\left[b_j-ja_j\right]z^j.\end{aligned} \tag{2.6}$$

Squaring the moduli of both sides of (2.6) and integrating along $|z| = r < 1$ and on using the fact that $|\psi(z)| \leq \beta|z|$, we obtain

$$\begin{aligned}&\sum_{j=2}^{m}\left[\left[1+\lambda(j-1)\right]C(n,j)(|ja_j-b_j|)\right]^2 r^{2j}\\&<\beta^2\left[(\alpha+1)^2r^2+\sum_{j=2}^{m-1}\left[\left[1+\lambda(j-1)\right]C(n,j)(|\alpha ja_j+b_j|)\right]^2 r^{2j}\right]\end{aligned}$$

Letting $r \to 1$ on the last inequality, we obtain

$$\begin{aligned}&\sum_{j=2}^{m}\left[\left[1+\lambda(j-1)\right]C(n,j)(|ja_j-b_j|)\right]^2\\&<\beta^2\left[(\alpha+1)^2+\sum_{j=2}^{m-1}\left[\left[1+\lambda(j-1)\right]C(n,j)(|\alpha ja_j+b_j|)\right]^2\right.\end{aligned}$$

This implies that

$$\begin{aligned}
&\Big[\big[1+\lambda(m-1)\big]C(n,m)(|ma_m-b_m|)\Big]^2\\
\le\ & \beta^2(1+\alpha)^2+\beta^2\sum_{j=2}^{m-1}\Big[\big[1+\lambda(j-1)\big]C(n,j)(|\alpha j a_j+b_j|)\Big]^2\\
&-\sum_{j=2}^{m-1}\Big[\big[1+\lambda(j-1)\big]C(n,j)(|ja_j-b_j|)\Big]^2\\
\le\ & \beta^2(1+\alpha)^2+(\beta^2\alpha^2-1)\sum_{j=2}^{m-1}\Big[\big[1+\lambda(j-1)\big]C(n,j)(|\alpha j a_j+b_j|)\Big]^2 j^2|a_j|^2\\
&+(\beta^2-1)\sum_{j=2}^{m-1}\Big[\big[1+\lambda(j-1)\big]C(n,j)(|ja_j-b_j|)\Big]^2|b_j|^2\\
&+2\alpha\beta^2\sum_{j=2}^{m-1}j\big[1+\lambda(j-1)\big]C(n,j)|a_j||b_j|+2\sum_{j=2}^{m-1}j\big[1+\lambda(j-1)\big]C(n,j)|a_j||b_j|.
\end{aligned}$$

Then

$$\begin{aligned}
&\Big[\big[1+\lambda(m-1)\big]C(n,m)(|ma_m-b_m|)\Big]^2\\
\le\ & 2\alpha\beta^2\sum_{j=2}^{m-1}j\big[1+\lambda(j-1)\big]C(n,j)|a_j||b_j|+2\sum_{j=2}^{m-1}j\big[1+\lambda(j-1)\big]C(n,j)|a_j||b_j|,
\end{aligned}$$

$(|a_1|=|b_1|=1)$, since $0\le\alpha\le 1, 0<\beta\le 1$, $k\ge 1$, $\lambda\ge 0$ and $n\in\mathbb{N}_0$.

First, we give two meaningful conclusions about the class $K_S^{(k)}(n,\lambda,\alpha,\beta)$.

Theorem 2.3 *The function* $f\in K_S^{(k)}(n,\lambda,\alpha,\beta)$ *if and only if*

$$\frac{z(D^n f(z))'}{D^n f_k(z)}\prec\frac{1+\beta z}{1-\alpha\beta z},$$

where $\prec$ *denotes the subordination between analytic functions (see [2]).*

Proof. Let $f\in K_S^{(k)}(n,\lambda,\alpha,\beta)$. Then from (1.12) we have

$$\left|\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f_k(z)}-1\right|^2<\beta^2\left|\frac{\alpha z(D_\lambda^n f(z))'}{D_\lambda^n f_k(z)}+1\right|^2$$

expanding it we get

$$(1-\alpha^2\beta^2)\left|\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f_k(z)}\right|^2-2(1+\alpha\beta^2)\mathrm{Re}\Big\{\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f_k(z)}\Big\}<\beta^2-1$$

If $\alpha \neq 1$ or $\beta \neq 1$, we have

$$\left|\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f_k(z)}\right|^2 - 2\frac{1+\alpha\beta^2}{1-\alpha^2\beta^2}\mathrm{Re}\left\{\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f_k(z)}\right\} + \left(\frac{1+\alpha\beta^2}{1-\alpha^2\beta^2}\right)^2 < \frac{\beta^2-1}{1-\alpha^2\beta^2} + \left(\frac{1+\alpha\beta^2}{1-\alpha^2\beta^2}\right)^2,$$

that is,

$$\left|\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f_k(z)} - \frac{1+\alpha\beta^2}{1-\alpha^2\beta^2}\right|^2 < \frac{\beta^2(1+\alpha)^2}{(1-\alpha^2\beta^2)2},$$

then

$$\left|\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f_k(z)} - \frac{1+\alpha\beta^2}{1-\alpha^2\beta^2}\right| < \frac{\beta(1+\alpha)}{(1-\alpha^2\beta^2)},$$

That the value region of $G(z) = z(D_\lambda^n f(z))'/D_\lambda^n f_k(z)$ is contained int the disk whose center is $(1+\alpha\beta^2)/(1-\alpha^2\beta^2)$ and radius is $\beta(1+\alpha)/(1-\alpha^2\beta^2)$ maps the unit disk to the disk

$$\left|w - \frac{1+\alpha\beta^2}{1-\alpha^2\beta^2}\right| < \frac{\beta(1+\alpha)}{(1-\alpha^2\beta^2)}.$$

Note that $G(\mathbf{U}) \subset p(\mathbf{U})$, $G(0) = p(0)$, and $p(z)$ is univalent in $\mathbf{U}$, we obtain the conclusions

$$\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f_k(z)} \prec p(z) = \frac{1+\beta z}{1-\alpha\beta z}.$$

Conversely, let

$$\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f_k(z)} \prec \frac{1+\beta z}{1-\alpha\beta z},$$

then

$$\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f_k(z)} = \frac{1+\beta w(z)}{1-\alpha\beta w(z)}, \tag{2.7}$$

where $w(z)$ is analytic in $\mathbf{U}$, and $w(0) = 0$, $|w(z)| < 1$. By simple calculation we can easily obtain from (2.7) that

$$\left|\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f_k(z)} - 1\right| < \beta\left|\frac{\alpha z(D_\lambda^n f(z))'}{D_\lambda^n f_k(z)} + 1\right|,$$

that is, $f \in K_S^{(k)}(n, \lambda, \alpha, \beta)$.
If $\alpha = \beta = 1$, inequality (1.12) becomes

$$\left|\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f_k(z)} - 1\right| < \left|\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f_k(z)} + 1\right|.$$

It is clear that $(z(D_\lambda^n f(z))'/D_\lambda^n f_k(z)) \prec (1+z)/(1-z)$. The proof of Theorem 2.3 is complete.

Remark 2.4 *From Theorem 2.3 we have*

$$Re\Big\{\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f_k(z)}\Big\} > 0, \qquad z \in \boldsymbol{U}, \tag{2.8}$$

because of $Re\Big\{(1+\beta z)/(1-\alpha\beta z)\Big\} > 0.$

Theorem 2.5 *Let* $f \in K_S^{(k)}(n, \lambda, \alpha, \beta)$*, then* $f_k \in S^*$.

Proof. Suppose that $f \in K_S^{(k)}(n, \lambda, \alpha, \beta)$, substituting z by $\varepsilon^\mu z$ in (2.8) respectively ($\mu = 0, 1, 2, ..., k-1$; $\varepsilon^k = 1$), then (2.8) is also true that is,

$$\mathrm{Re}\left\{\frac{\varepsilon^\mu z(D_\lambda^n f(\varepsilon^\mu z))'}{D_\lambda^n f_k(\varepsilon^\mu z)}\right\} > 0, \qquad (\mu = 0, 1, 2, ..., k-1). \tag{2.9}$$

According to the definition of $f_k(z)$ and $\varepsilon^k = 1$, we know $f_k(\varepsilon^\mu z) = \varepsilon^\mu f_k(z)$. Let $\mu = 0, 1, 2, ..., k-1$ in (2.9) respectively, and obviously we can get

$$\mathrm{Re}\left\{\frac{1}{k}\sum_{\mu=0}^{k-1}\frac{\varepsilon^\mu z(D_\lambda^n f(\varepsilon^\mu z))'}{\varepsilon^\mu D_\lambda^n f_k(z)}\right\} = \mathrm{Re}\Big\{\frac{z(D_\lambda^n f_k(z))'}{D_\lambda^n f_k(z)}\Big\} > 0, \tag{2.10}$$

that is $f_k \in S^*$.

Remark 2.6 *From Theorem 2.5 and inequality (2.8), we know that if* $f \in K_S^{(k)}(n, \lambda, \alpha, \beta)$*, then* f *is a close to convex function. So* $K_S^{(k)}(n, \lambda, \alpha, \beta)$ *is a subclass of the class belongs to close-to-convex functions.*

Theorem 2.7 *Let* $f \in K_S^{(k)}(n, \lambda, \alpha, \beta)$*, then*
(i) For $l \geq 2$,

$$\Big[\big[1+\lambda k(l-1)\big]C(n, k(l-1)+1)(l-1)k\Big]^2|a_{(l-1)k+1}|^2$$

$$\leq 2(\alpha\beta^2+1)\sum_{\delta=1}^{l-1}((\delta-1)k+1)\big[1+\lambda k(\delta-1)\big]C(n, k(i-1)+1)|a_{k(\delta-1)+1}|^2$$

$$(|a_1| = 1).$$

(ii) For $m \geq 2$, $m \neq (l-1)k+1$,

$$\Big[\big[1+\lambda(m-1)\big]C(n, m)m\Big]^2|a_m|^2$$

$$\leq 2(\alpha\beta^2+1) \sum_{\delta=1}^{[\frac{m-2}{k}+1]} ((\delta-1)k+1)\big[1+\lambda k(\delta-1)\big]C(n,k(i-1)+1)|a_{k(\delta-1)+1}|^2$$

$$(|a_1|=1).$$

where $[\frac{m-2}{k}+1]$ *denote the biggest integer* $\leq \frac{m-2}{k}+1$.

Proof. By definition of $f_k(z)$ we have

$$\begin{aligned} f_k(z) &= \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon^{-\nu} f(\varepsilon^{\nu} z) \\ &= \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon^{-\nu}\Big[\varepsilon^{\nu} z + \sum_{m=2}^{\infty} a_m(\varepsilon^{\nu} z)^m\Big] \\ &= z + \sum_{l=2}^{\infty} a_{(l-1)k+1} z^{(l-1)k+1}. \end{aligned}$$

and

$$D_\lambda^n f_k(z) = z + \sum_{l=2}^{\infty} [1+\lambda k(l-1)]C(n,(l-1)k+1)a_{(l-1)k+1}z^{(l-1)k+1}.$$

For $f_k(z) \in S^*$, $f(z)$ and $f_k(z)$ satisfy the condition of Lemma 2.2. By using Lemma 2.2, let $m=(l-1)k+1$ in (2.3) we get (i) of Theorem 2.7. If $m \neq (l-1)k+1$, $m \geq 2$ from 2.3 we obtain (ii) of Theorem 2.7.

3 The Integral Representation

In this section, we give representation of functions belonging in the class $K_S^{(k)}(n,\lambda,\alpha,\beta)$.

Theorem 3.1 *Let* $f \in K_S^{(k)}(n,\lambda,\alpha,\beta)$, *then we have*

$$D_\lambda^n f_k(z) = z.exp\left\{\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^{\mu} z} \frac{(1+\alpha)\beta w(t)}{t[1-\alpha\beta w(t)]}dt\right\},$$

where $D_\lambda^n f_k(z)$ *is defined by (1.11),* $w(z)$ *is analytic in* $\boldsymbol{U}$ *and* $w(0)=0$, $|w(z)| < 1$.

Proof. Suppose that $f \in K_S^{(k)}(n,\lambda,\alpha,\beta)$, from Theorem 2.3 we have

$$\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f_k(z)} = \frac{1+\beta w(z)}{1-\alpha\beta w(z)},$$

where $w(z)$ is analytic in $\mathbf{U}$, and $w(0) = 0$, $|w(z)| < 1$. Substituting z by $\varepsilon^\mu z$ in this equality respectively ($\mu = 0, 1, 2, ..., k-1$; $\varepsilon^k = 1$), and using the same method in Theorem 2.5 we get

$$\frac{z(D_\lambda^n f_k(z))'}{D_\lambda^n f_k(z)} = \frac{1}{k}\sum_{\mu=0}^{k-1} \frac{1+\beta w(\varepsilon^\mu z)}{1-\alpha\beta w(\varepsilon^\mu z)},$$

then we get

$$\frac{(D_\lambda^n f_k(z))'}{D_\lambda^n f_k(z)} - \frac{1}{z} = \frac{1}{k}\sum_{\mu=0}^{k-1} \frac{(1+\alpha)\beta w(\varepsilon^\mu z)}{z[1-\alpha\beta w(\varepsilon^\mu z)]}.$$

Integrating this equality we have

$$\begin{aligned}\log\left\{\frac{D_\lambda^n f_k(z)}{z}\right\} &= \frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^z \frac{(1+\alpha)\beta w(\varepsilon^\mu \zeta)}{\zeta[1-\alpha\beta w(\varepsilon^\mu \zeta)]}d\zeta \\ &= \frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu z} \frac{(1+\alpha)\beta w(t)}{\zeta[1-\alpha\beta w(t)]}dt,\end{aligned}$$

that is

$$D_\lambda^n f_k(z) = z.\exp\left\{\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu z} \frac{(1+\alpha)\beta w(t)}{t[1-\alpha\beta w(t)]}dt\right\},$$

the proof of Theorem 3.1 is complete.

Theorem 3.2 *Let $f \in K_S^{(k)}(n, \lambda, \alpha, \beta)$, then we have*

$$D_\lambda^n f(z) = \int_0^z \frac{1+\beta w(\zeta)}{1-\alpha\beta w(\zeta)}.exp\left\{\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu \zeta} \frac{(1+\alpha)\beta w(t)}{t[1-\alpha\beta w(t)]}dt\right\}d\zeta,$$

where $D_\lambda^n f(z)$ is defined by (1.4), $w(z)$ is analytic in $\boldsymbol{U}$ and $w(0) = 0$, $|w(z)| < 1$.

Proof. From Theorem 3.1 we have

$$\begin{aligned}(D_\lambda^n f(z))' &= \frac{D_\lambda^n f_k(z)}{z}.\frac{1+\beta w(z)}{1-\alpha\beta w(z)} \\ &= \frac{1+\beta w(z)}{1-\alpha\beta w(z)}.\exp\left\{\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu z} \frac{(1+\alpha)\beta w(t)}{t[1-\alpha\beta w(t)]}dt\right\},\end{aligned}$$

by integrating this equality we can obtain the conclusion of Theorem 3.2.

4 Sufficient Condition

At last, we give sufficient condition of functions belonging to the class $K_S^{(k)}(n,\lambda,\alpha,\beta)$.

Theorem 4.1 *Let the function f be defined by (1.1). Then $f \in K_S^{(k)}(n,\lambda,\alpha,\beta)$ if and only if*

$$\sum_{m=1}^{\infty}[1+\lambda(m-1)]C(n,m)[(1+\alpha\beta)(mk+1)+\beta-1]|a_{mk+1}| + \sum_{\substack{m=2\\ m\neq lk+1}}^{\infty}[1+\lambda(m-1)]C(n,m)(1+\alpha\beta)m|a_m| < (1+\alpha)\beta \quad (4.1)$$

where $0 \le \alpha \le 1$ *and* $0 < \beta \le 1$.

Proof. Suppose that f be defined by (1.1), then for $|z| = r < 1$ we have

$$\begin{aligned}
&\Big|z(D_\lambda^n f(z))' - D_\lambda^n f_k(z)\Big| - \beta\Big|\alpha z(D_\lambda^n f(z))' + D_\lambda^n f_k(z)\Big| \\
&= \Big|z + \sum_{m=2}^{\infty} m[1+\lambda(m-1)]C(n,m)a_m z^m - z - \sum_{m=2}^{\infty}[1+\lambda(m-1)]C(n,m)a_m b_m z^m\Big| \\
&\quad -\beta\Big|\alpha z + \sum_{m=2}^{\infty}\alpha m[1+\lambda(m-1)]C(n,m)a_m z^m + z + \sum_{m=2}^{\infty}[1+\lambda(m-1)]C(n,m)a_m b_m z^m\Big|,
\end{aligned}$$

where

$$b_m = \frac{1}{k}\sum_{\nu=0}^{k-1}\varepsilon^{(m-1)\nu}, \qquad \varepsilon^k = 1. \tag{4.2}$$

Thus we have

$$\begin{aligned}
&\Big|z(D_\lambda^n f(z))' - D_\lambda^n f_k(z)\Big| - \beta\Big|\alpha z(D_\lambda^n f(z))' + D_\lambda^n f_k(z)\Big| \\
&\le \Big|z + \sum_{m=2}^{\infty}[1+\lambda(m-1)]C(n,m)(m-b_m)|a_m|r^m - \beta[(1+\alpha)r \\
&\quad - \sum_{m=2}^{\infty}[1+\lambda(m-1)]C(n,m)(\alpha m + b_m)|a_m|r^m] \\
&< r\Big\{-(1+\alpha)\beta + \sum_{m=2}^{\infty}[1+\lambda(m-1)]C(n,m)[(m-b_m)+\beta(\alpha m+b_m)]C(n,m)|a_m|\Big\}.
\end{aligned}$$

From the definition of b_m we know

$$b_m = \begin{cases} 1, & m = lk+1, \\ 0, & m \neq lk+1, \end{cases}$$

substituting it into last inequality, we get

$$\Big|z(D^n_\lambda f(z))' - D^n_\lambda f_k(z)\Big| - \beta\Big|\alpha z(D^n_\lambda f(z))' + D^n_\lambda f_k(z)\Big|$$

$$< \; r\Bigg\{ -(1+\alpha)\beta + \sum_{m=1}^{\infty}[1+\lambda(m-1)]C(n,m)[(1+\alpha\beta)(mk+1)+\beta-1]|a_{mk+1}|$$

$$+ \sum_{\substack{m=2\\ m\neq lk+1}}^{\infty} [1+\lambda(m-1)]C(n,m)(1+\alpha\beta)m|a_m|\Bigg\} < 0.$$

The proof of the theorem is complete.

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