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Associated Prime Ideals of a Complete Lattice

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Abstract. In this article, the annihilator of a subset of a complete distributive lattice L is defined. Annihilator primes and associated primes of a lattice L are defined. It is shown that Annihilator of any subset of a complete chain is an associated prime.

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1. Introduction

Through out this article, $(L, \leq)$ denotes a lattice, unless other wise stated. For any $a, b \in L$, $a \wedge b$ denotes the infimum of $\{a, b\}$ and $a \vee b$ denotes the supremum of $\{a, b\}$. For a complete lattice $(L, \leq)$, 0 denotes the infimum of L. Henceforth, we write L for $(L, \leq)$. For the definition of a lattice, its types, examples of lattice, bounds of subsets of a lattice, infimum and supremum of subsets of a lattice, and the basic results, the reader is referred to chapter 12 of Liu [2].

A highly relevant class of lattices are the complete lattices. A lattice is complete if all of its subsets have both a supremum and an infimum. In mathematics, a complete lattice is a partially ordered set in which all subsets have both a supremum and an infimum. Note that in the special case where A is the empty set the infimum of A will be the greatest element of L. Likewise, the supremum of the empty set yields the least element. Since the definition also assures the existence of binary meets and joins, complete lattices do thus form a special class of bounded lattices. Complete lattices appear in many applications in mathematics and computer science.

This article concerns prime ideals (more particularly associated prime ideals) of a complete lattice. Recall that Boolean Prime Ideal Theorem states that ideals in a Boolean Algebra can be extended to prime ideals. A variation of this statement for filters on sets is known as the Ultrafilter Lemma. Recall that a filter on a set X is a collection of nonempty subsets of X that is closed under finite intersection and under superset. An ultrafilter is a maximal filter. The ultrafilter lemma states that every filter on a set X is a subset of some ultrafilter on X (a maximal filter of nonempty subsets of X.) This lemma is most often used in the study of topology.

We define the annihilator of a subset of a complete distributive lattice L. We also define a fully faithful lattice and show that a chain, which is also complete, is fully faithful. This is proved in Proposition (2.10). Annihilator primes and associated primes of a lattice L are defined. It is shown that annihilator of any subset of a complete chain is an associated prime. This is proved in Theorem (2.20).

2. Associated primes of a complete lattice

Definition 2.1. Let L be a complete lattice, which is also distributive. Let A be a non empty subset of L. Annihilator of A is denoted by Ann(A), and is defined as: Ann(A) $= \{l \in L$, such that $a \wedge l = 0$, for all $a \in A\}$.

Example 2.2. *Let $L = \{1, 3, 9, 27\}$. L is a complete, distributive lattice with respect to divisibility. Let $A = \{3, 9\}$. Then Ann(A) $= \{1\}$. Let $K = \{1, 2, 3, 4, 6, 8, 12, 24\}$, and $B = \{3\}$. Then Ann(B) $= \{1, 2, 4, 8\}$.*

Example 2.3. *Let $X = \{a, b, c\}$. Now P(X), the power set of X is a complete distributive lattice with respect to set inclusion . Let $A = \{\ \{a\}, \{a, b\}\}$. Then Ann(A) $= \{\ \{c\}\}$.*

Proposition 2.4. *Let L be a complete lattice, which is also distributive. Let $A \subseteq B$ be two non empty subsets of L. Then $Ann(B) \supseteq Ann(A)$.*

Proof. Let $l \in Ann(B)$. Then $b \wedge l = 0$, for all $b \in B$. Now let $a \in A$. Then $a \in B$, and therefore $a \wedge l = 0$. So $l \in Ann(A)$. Hence $Ann(B) \supseteq Ann(A)$. □

Definition 2.5. Let L be a complete lattice, which is also distributive. A subset $B \subseteq L$ is said to be faithful if Ann(B) $= \{0\}$.

Example 2.6. *Let $L = \{1, 2, 3, 6, 10, 15, 30\}$ with respect to divisibility. Then $B = \{1, 3, 6\}$ is a faithful subset of L, but $C = \{1, 3\}$ is not faithful, as Ann(C) $= \{1, 2, 10\}$.*

Proposition 2.7. *Let L be a complete lattice, which is also distributive. Let A be a non empty subset of L. Then Ann(A) is an ideal of L.*

Proof. $Ann(A) \neq \phi$, as $0 \in Ann(A)$. Now any $r, s \in Ann(A)$ imply $r \wedge a = 0$ and $s \wedge a = 0$ for all $a \in A$. Now for all $a \in A$, $(r \vee s) \wedge a = (r \wedge a) \vee (s \wedge a)$ $= 0 \vee 0 = 0$. Therefore $(r \vee s) \in Ann(A)$.
Now let $r \in Ann(A)$, and $l \in L$. We have $r \wedge a = 0$ for all $a \in A$, and $(r \wedge l) \wedge a$ $= (l \wedge r) \wedge a = l \wedge (r \wedge a) = l \wedge 0 = 0$. Therefore $(r \wedge l) \in Ann(A)$. Hence Ann(A) is an ideal of L. □

Definition 2.8. Let L be a complete lattice, which is also distributive. We say that L is a fully faithful lattice if Ann(A) = 0, where A is any non empty subset of L.

Example 2.9. *In Example (2.2) above, L is fully faithful.*

Proposition 2.10. *Every chain, which is also complete, is a fully faithful lattice.*

Proof. Let L be a chain and $\{0\} \neq A$ be a non empty subset of L. Let $x \in Ann(A)$. Then $x \wedge a = 0$ for all $a \in A$. Now L is a chain, therefore $x \leq a$ for at least one $0 \neq a \in A$, or $a \leq x$ for all $a \in A$. If $a \leq x$ for all $a \in A$, then $x \wedge a =$ a, which implies that a = 0 for all $a \in A$, a contradiction. So $x \leq a$ for at least one $0 \neq a \in A$, and therefore $x \wedge a =$ x. Thus x = 0. Therefore Ann(A) = 0. Hence L is a fully faithful lattice. □

Recall that an ideal P of a lattice L is called a prime ideal if $P \neq L$, and if for any $a, b \in L$ with $(a \wedge b) \in P$, we have $a \in A$ or $b \in A$.

Definition 2.11. Let L be a lattice. An ideal P is said to be a minimal prime ideal if P is a prime ideal and does not contain properly a prime ideal.

Example 2.12. *Consider $L = \{1, 2, 5, 10\}$. L is a lattice with respect to divisibility. $P = \{2\}$ is a minimal prime ideal. $S = \{1, 2\}$ is a prime ideal, but is not a minimal prime ideal, as $\{2\} \subset \{1, 2\}$, and $\{2\}$ is a prime ideal.*

Proposition 2.13. *In a lattice, every ideal contains a minimal prime ideal.*

Proof. The proof is on the same lines as in Proposition (2.3) of Goodearl and Warfield [1], in which the case of existence of minimal prime ideals of a ring has been proved. □

Definition 2.14. Let L be a complete distributive lattice. An ideal A of L is called an annihilator prime if A is a prime ideal and is also annihilator of some non empty subset $\{0\} \neq B$ of L.

Example 2.15. *Let $L = \{1, 2, 5, 10\}$. L is complete distributive lattice with respect to divisibility. $A = \{1, 2\}$ is a prime ideal. Now let $M = \{5\}$. Then A = Ann(M). Therefore A is an annihilator prime.*
Let $B = \{1\}$, and $C = \{2, 5\}$. Then Ann(C) = B. Now B is not a prime ideal, as $2 \wedge 5 = 1 \in P$, but $2 \notin B$ and $5 \notin B$. Therefore B is not an annihilator prime.

Example 2.16. *In Example (2.3) above, Ann(A) is not a prime as* $\{b, c\} \cap \{a, c\} \in Ann(A) = \{\{c\}\}$*, but* $\{b, c\} \notin Ann(A)$ *and* $\{a, c\} \notin Ann(A)$*.*

Definition 2.17. Let L be a complete distributive lattice. An ideal A of L is called an associated prime if A is a prime ideal and is annihilator of some non empty subset $\{0\} \neq B$ of L. Further more A is also the annihilator of any non empty subset C of B. It is clear that an associated prime is an annihilator prime, but every annihilator prime need not be an associated prime.

Example 2.18. *Let* $L = \{1, 3, 5, 7, 35\}$ *with respect to divisibility. Let* $B = \{5, 7, 35\}$*, and* $C = \{5\}$*. Then Ann(B)* $= \{1, 3\}$*, and Ann(C)* $= \{1, 3, 7\}$*. Now* $\{1, 3\}$ *is a prime ideal of L, so it is an annihilator prime, but not an associated prime.*

Example 2.19. *Let* $L = \{1, 2, 4, 5, 10, 20\}$*. With respect to divisibility, L is a complete distributive lattice. Let* $A = \{1, 5\}$*, and* $B = \{2, 4\}$*. Then* $A =$ *Ann(B), and* $A =$ *Ann(C) for any subset C of B. More over A is a prime ideal. Therefore A is an associated prime ideal.*

Theorem 2.20. *In a chain, which is also complete; annihilator of any non-zero subset is an associated prime ideal.*

Proof. Let L be a chain, which is also complete. Let $\{0\} \neq B$ be a non empty subset of L. Let Ann(B) = A. Now A is an ideal by Proposition (2.7). We will show that A is a prime ideal. Let $a, b \in L$ be such that $(a \wedge b) \in A$. Now L is a chain, therefore $b \leq a$, or $a \leq b$. If $a \leq b$, then $a \wedge b =$ a, and so $a \in A$. If $b \leq a$, then $a \wedge b =$ b, and so $b \in A$. Therefore A is a prime ideal.
Let $0 \neq C \subseteq B$ be a non empty subset of B. Then by Proposition (2.4), $Ann(B) \subseteq Ann(C)$. We will show that A = Ann(C). Suppose not, and $0 \neq b \in Ann(C)$, $b \notin A$. Now $b \wedge c = 0$, for all $c \in C$. Now L is a chain, therefore $b \leq c$ for at least one $c \in C$, or $c \leq b$ for all $c \in C$. If $b \leq c$ for at least one $c \in C$, then b $= b \wedge c = 0$, a contradiction. If $c \leq b$ for all $c \in C$, then c $= b \wedge c = 0$. Therefore C = $\{0\}$, again a contradiction. So our supposition must be wrong. Therefore A = Ann(C), and hence A is an associated prime ideal of L. □

References

[1] K.R. Goodearl and R.B. Warfield Jr., An introduction to non-commutative Noetherian rings. Cambridge University Press, 1989.

[2] C. L. Liu, Elements of Discrete Mathematics. Tata Mcgraw Hill, 2000.

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