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Spheres in an Euclidean Space

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Abstract

Let M be an n-dimensional orientable compact hypersurface in an $(n+1)$-dimensional Euclidean space R^{n+1}, $n \geq 2$. If the gradient $\nabla\alpha$ of the mean curvature α and the scalar curvature S of the hypersurface M satisfy the inequality

$$Sk_0 \leq n(n-1)k_0\alpha^2 - (n-1)\left\|\nabla\alpha\right\|^2$$

where k_0 is the infimum of the sectional curvatures of the hypersurface, then it is shown that α is a constant and M is the sphere in $S^n(\alpha)$.

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1 Introduction

The class of compact hypersurfaces in the Euclidean space R^{n+1} is quite large and therefore it is an interesting question in Geometry to obtain conditions which characterize spheres in this class. This question has been of considerable interest to many geometers and had been approached using various invariants of the hypersurfaces. Most natural invariants of a hypersurface are the mean curvature, Ricci curvature and scalar curvature. Nomizu and Smyth have studied non-negatively curved hypersurfaces with constant mean curvature in real space form and in particular they have shown that such compact hypersurfaces in a Euclidean space are spheres (cf. [10], [11]). Hypersurfaces with constant mean curvature and higher order mean curvatures in a real space form have also been studied by Chen, Montiel-Ros, Ripoll, Ros and obtained different characterizations for extrinsic spheres (cf. [2], [3], [4], [9], [12], [14]). Similarly Ros [13] has studied compact embedded hypersurfaces with constant scalar curvature in an Euclidean space and proved that they are essentially spheres. In this paper we use the invariants α the mean curvature and S the scalar curvature of the hypersurface to characterize the spheres. An interesting question is, using the invariants α, S of the hypersurface, how to characterize extrinsic spheres in the Euclidean space R^{n+1}? The motivation of this question comes from the following: A sphere $S^n(c)$ in R^{n+1}, satisfies the equality

$$cS = n(n-1)c^2 - (n-1)\left\|\nabla\alpha\right\|^2$$

$\nabla\alpha$ being the gradient of the mean curvature α (This follows from the Gauss equation for the hypersurface in a Euclidean space and the fact that $\alpha = -\sqrt{c}$ is constant for the sphere $S^n(c)$) . This raises a question, does a compact hypersurface of R^{n+1} satisfying above equality necessarily a sphere? In this paper we show that the answer to this question is in affirmative, and indeed we prove the following result which gives a characterization of spheres in R^{n+1}.

Theorem 1.1 *Let M be an n-dimensional orientable compact hypersurface of the Euclidean space R^{n+1}, $n \geq 2$ and k_0 be the infimum of the sectional curvatures of M. If the scalar curvature S and the mean curvature α of M satisfy*

$$Sk_0 \leq n(n-1)k_0\alpha^2 - (n-1)\left\|\nabla\alpha\right\|^2$$

then α is a constant and M is the sphere $S^n(\alpha)$.

We note that in above theorem we do not require the sectional curvature of the hypersurface to be non-negative.

2 Preliminaries

Let M be an orientable hypersurface of the Euclidean space R^{n+1}. We denote the induced metric on M by g. Let $\overline{\nabla}$ be the Euclidean connection on R^{n+1} and ∇ be the Riemannian connection on M with respect to the induced metric g. Let N be the unit normal vector field and A be the shape operator of M. Then the Gauss and Weingarten formulas for the hypersurface are (cf. [1])

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \overline{\nabla}_X N = -AX, \qquad X, Y \in \mathfrak{X}(M) \qquad (2.1)$$

where $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M. We also have the following Gauss and Codazzi equations

$$R(X, Y)Z = g(AY, Z)AX - g(AX, Z)AY \qquad (2.2)$$

$$(\nabla A)(X, Y) = (\nabla A)(Y, X), \qquad X, Y \in \mathfrak{X}(M) \qquad (2.3)$$

where R is the curvature tensor field of the hypersurface and $(\nabla A)(X, Y) = \nabla_X AY - A\nabla_X Y$. The mean curvature α of the hypersurface is given by $n\alpha = \sum_i g(Ae_i, e_i)$, where $\{e_1, .., e_n\}$ is a local orthonormal frame on M. The square of the length of the shape operator A is given by

$$\|A\|^2 = \sum_{ij} g(Ae_i, e_j)^2 = trA^2$$

From equation (2.2) we get the following expression for the Ricci tensor field

$$Ric(X, Y) = n\alpha g(AX, Y) - g(AX, AY) \tag{2.4}$$

The scalar curvature S of the hypersurface is given by

$$S = n^2\alpha^2 - \|A\|^2 \tag{2.5}$$

3 Some Lemmas

Let M be an orientable hypersurface of the Euclidean space R^{n+1} and $\nabla\alpha$ be the gradient of the mean curvature function α. Then we have

Lemma 3.1 *Let M be an n-dimensional orientable hypersurface of the Euclidean space R^{n+1} and $\{e_1, .., e_n\}$ be a local orthonormal frame on the hypersurface M. Then*

$$\sum_i (\nabla A)(e_i, e_i) = n\nabla\alpha$$

The proof is straightforward and follows from the symmetry of A and the equation (2.3).

Lemma 3.2 *Let M be an n-dimensional orientable compact hypersurface of the Euclidean space R^{n+1} and $\{e_1, .., e_n\}$ be a local orthonormal frame that diagonalizes A with $Ae_i = \lambda_i e_i$. Then*

$$\int_M \left(\sum_{i\neq j} (\lambda_i - \lambda_j)^2\right) dv = 2n \int_M \left(\|A\|^2 - n\alpha^2\right) dv$$

The proof follows from the fact that $\sum_i \lambda_i = n\alpha$ and $\sum_i \lambda_i^2 = \|A\|^2$.

Lemma 3.3 *Let M be an n-dimensional orientable hypersurface of the Euclidean space R^{n+1}, $n \geq 2$. Then*

$$\|\nabla A\|^2 \geq n \|\nabla\alpha\|^2$$

where $\|\nabla A\|^2 = \sum_{ij} \|(\nabla A)(e_i, e_j)\|^2$ for a local orthonormal frame $\{e_1, .., e_n\}$ on M, moreover the equality holds if and only if α is constant and A is parallel.

Proof. Define an operator $B : \mathfrak{X}(M) \to \mathfrak{X}(M)$ by $B = A - \alpha I$. Then we have

$$(\nabla B)(X, Y) = (\nabla A)(X, Y) - (X\alpha)Y$$

which gives

$$\begin{aligned}\|\nabla B\|^2 &= \|\nabla A\|^2 + n\|\nabla\alpha\|^2 - 2\sum_{ij} g\left((\nabla A)(e_i, e_j), e_j\right) g(\nabla\alpha, e_i)\\ &= \|\nabla A\|^2 + n\|\nabla\alpha\|^2 - 2\sum_{j} g\left(\nabla\alpha, (\nabla A)(e_j, e_j)\right)\\ &= \|\nabla A\|^2 - n\|\nabla\alpha\|^2\end{aligned}$$

This proves that $\|\nabla A\|^2 \geq n\|\nabla\alpha\|^2$. The equality holds if and only if $\nabla B = 0$ that is, $(\nabla A)(X, Y) = X(\alpha)Y$. Using Codazzi equation (2.3) and $n \geq 2$ we get that α is a constant and that $\nabla A = 0$.

Lemma 3.4 *Let M be an n-dimensional orientable compact hypersurface of the Euclidean space R^{n+1}. Then*

$$\int_M \left(\sum_i g(\nabla_{e_i}(\nabla\alpha), Ae_i)\right) dV = -n\int_M \|\nabla\alpha\|^2\, dV$$

where $\{e_1, .., e_n\}$ is a local orthonormal frame on M.

Proof. Choosing a point wise covariant constant local orthonormal frame $\{e_1, ..., e_n\}$ on M, we compute

$$\begin{aligned}div\,(A(\nabla\alpha)) &= \sum_i e_i g(\nabla\alpha, Ae_i) = \sum_i g(\nabla_{e_i}(\nabla\alpha), Ae_i) + \sum_i g(\nabla\alpha, (\nabla A)(e_i, e_i))\\ &= \sum_i g(\nabla_{e_i}(\nabla\alpha), Ae_i) + n\|\nabla\alpha\|^2\end{aligned}$$

Integrating this equation we get the Lemma.

We define the second covariant derivative $(\nabla^2 A)(X, Y, Z)$ as

$$(\nabla^2 A)(X, Y, Z) = \nabla_X(\nabla A)(Y, Z) - A(\nabla_X Y, Z) - A(Y, \nabla_X Z),$$

then using the Ricci identity we get

$$(\nabla^2 A)(X, Y, Z) - (\nabla^2 A)(Y, X, Z) = R(X, Y)AZ - AR(X, Y)Z \qquad (3.1)$$

4 Proof of the Theorem

Let M be an n-dimensional orientable compact hypersurface of the Euclidean space R^{n+1}. Define a function $f : M \to R$ by $f = \frac{1}{2}\|A\|^2$. Then by a straightforward computation we get the Laplacian Δf of the smooth function f as

$$\Delta f = \|\nabla A\|^2 + \sum_{ij} \left((\nabla^2 A)(e_j, e_j, e_i), Ae_i\right) \tag{4.1}$$

where $\{e_1, ..., e_n\}$ is local orthonormal frame on M.

Using the equation (2.3), we arrive at

$$g(\left(\nabla^2 A\right)(e_j, e_j, e_i), Ae_i) = g(\left(\nabla^2 A\right)(e_j, e_i, e_j), Ae_i)$$

Now using the Ricci identity (3.1) in above equation we get

$$\begin{aligned} g(\left(\nabla^2 A\right)(e_j, e_j, e_i), Ae_i) &= g(\left(\nabla^2 A\right)(e_i, e_j, e_j), Ae_i) + g(R(e_j, e_i)Ae_j, Ae_i) \\ &\quad - g(R(e_j, e_i)e_j, A^2 e_i) \end{aligned}$$

Thus in light of this equation the equation (4.1) takes the form

$$\begin{aligned} \Delta f &= \|\nabla A\|^2 + \sum_{ij} g(\left(\nabla^2 A\right)(e_i, e_j, e_j), Ae_i) \\ &\quad + \sum_{ij} \left[g(R(e_j, e_i)Ae_j, Ae_i) - g(R(e_j, e_i)e_j, A^2 e_i)\right] \end{aligned} \tag{4.2}$$

Using Lemma 3.1, we get

$$\sum_i \left(\nabla^2 A\right)(e_i, e_j, e_j) = n\nabla_{e_i}(\nabla\alpha). \tag{4.3}$$

Now we use local orthonormal frame $\{e_1, ..., e_n\}$ that diagonalizes A with $Ae_i = \lambda_i e_i$ to compute the sum as

$$\begin{aligned} \sum_{ij} \left[g(R(e_j, e_i)Ae_j, Ae_i) - g(R(e_j, e_i)e_j, A^2 e_i)\right] &= \sum_{i \neq j} \left[\lambda_i^2 - \lambda_i \lambda_j\right] K_{ij} \\ &= \frac{1}{2} \sum_{i \neq j} (\lambda_i - \lambda_j)^2 K_{ij} \end{aligned}$$

where K_{ij} is the sectional curvature of the plane section spanned by $\{e_i, e_j\}$. Using this last equation together with (4.3) in (4.2), we arrive at

$$\Delta f = \|\nabla A\|^2 + n\sum_i g(\nabla_{e_i}(\nabla\alpha), Ae_i) + \frac{1}{2}\sum_{i \neq j} (\lambda_i - \lambda_j)^2 K_{ij}$$

Integrating this equation and using Lemma 3.4 we arrive at

$$\int_M \left\{ \left[\|\nabla A\|^2 - n \|\nabla \alpha\|^2 \right] - n(n-1) \|\nabla \alpha\|^2 + \frac{1}{2} \sum_{i \neq j} (\lambda_i - \lambda_j)^2 K_{ij} \right\} dV = 0$$

Since $K_{ij} \geq k_0$, and $(\lambda_i - \lambda_j)^2 \geq 0$ the above equation together with equation (2.5) and Lemma 3.3, takes the form

$$\int_M \left\{ \left[\|\nabla A\|^2 - n \|\nabla \alpha\|^2 \right] + n \left(n(n-1)\alpha^2 k_0 - (n-1) \|\nabla \alpha\|^2 - Sk_0 \right) \right\} dV \leq 0$$

The condition in the statement of the theorem together with Lemma 3.3 and above inequality yields

$$\|\nabla A\|^2 = n \|\nabla \alpha\|^2 \tag{4.4}$$

$$Sk_0 = n(n-1)\alpha^2 k_0 - (n-1) \|\nabla \alpha\|^2 \tag{4.5}$$

Thus from equation (4.4) together with Lemma 3.4 we get that α is a constant and the shape operator A is parallel. Then by Theorem 4 in [8] we see that M being compact is a sphere $S^n(r)$ for some constant r. As α is a constant and in this case the infimum $k_0 = r \neq 0$, the equation (4.5) gives $S = n(n-1)\alpha^2$ that is $r = \alpha$ and consequently M is $S^n(\alpha)$.

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