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Characterization of Monotonically T_2 Spaces[1]

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Abstract

In this paper we shall give some characterizations of monotonically T_2-spaces, and a characterization of Hausdorff spaces.

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1 Introduction

R. E. Buck introduced *monotonically T_2 spaces* (also called *monotonically Hausdorff spaces*) in [2]. He proved that monotonic T_2 property is hereditary and that the box product of monotonically T_2 spaces is monotonically T_2. He also proved that every monotonically T_2-space is T_3 and that the converse is true for first countable spaces. Thus, Niemytzki's tangent disk space (also known as the Moore plane) is monotonically T_2, and any non-regular space is not monotonically T_2.

In this paper, we shall give some characterizations of monotonically Hausdorff spaces, and a new characterization of Hausdorff spaces in the same vein. Moreover, we shall prove that the topological property "monotonically Hausdorff" is a strongly properly open hereditary property.

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2 Preliminary Notes

Definition 2.1 *A topological space* $(X,\,\tau)$ *is called monotonically* T_2 *(or monotonically Hausdorff) if there is a function* $g:(X\times X)\backslash\Delta\rightarrow\tau$ *assigning to each ordered pair* $(x,\,y)$ *of distinct points in* X *an open neighborhood* $g(x,\,y)$ *of* x *such that:*

$$g(x,\,y)\cap g(y,\,x)=\phi; \tag{1}$$

$$\text{for each } M\subseteq X,\ \text{if } x\in\overline{\cup\{g(y,\,x):y\in M\}},\ \text{then } x\in\overline{M}. \tag{2}$$

We call such a function g *a monotone* T_2 *operator (or a monotone Hausdorff operator) on* X*. If, in addition, the function* g *satisfies the following condition:*

$$\text{if } z\in g(x,y),\ \text{then } g(z,y)\subseteq g(x,y), \tag{3}$$

then, the topological space (X,τ) *is called strongly monotonically* T_2 *(or strongly monotonically Hausdorff), and the function* g *is called a strong monotone* T_2 *operator (or a strong monotone Hausdorff operator) on* X*. Also, if the function* g *is satisfying the conditions (1), (2) and in addition the following condition:*

$$\text{if } x_0,\,x_1,\ldots,\,x_{n-1} \text{ are distinct points in } X \text{ and } x_n=x_0,\ \text{then } \bigcap_{i=0}^{n-1} g(x_i,x_{i+1})=\phi. \tag{4}$$

We call such a function g *an acyclic monotone* T_2 *operator (or an acyclic monotone Hausdorff operator) on* X*.*

Now, it is natural to ask the following questions:

Question 1. *Are the topological properties monotonically Hausdorff and strongly monotonically Hausdorff equivalent?*

Question 2. *Are the topological properties monotonically Hausdorff and acyclically monotonically Hausdorff equivalent?*

Also, we may ask the following question:

Question 3. *Is it possible to strengthen (weaken) the conditions in Definition 2.1, to obtain stronger (weaker) classes of spaces?*

F. Arenas [1] introduced the property so-called "*properly hereditary property*". Any topological property satisfying the condition (5) below, is called *properly (open) hereditary property.*

$$\textit{If every proper (open) subspace has the property, the whole space has the property} \tag{5}$$

Moreover, if the property is hereditary for open subspaces, and every proper open subspace has the property, then the whole space has the property, we call such a property *strongly properly open hereditary.*

3 Main Results

We shall divide this section into three subsection.

3.1 Properly Hereditary Property

The topological property "monotonically Hausdorff property" is properly open hereditary property, more precisely, we have the following result.

Theorem 3.1 *Let (X, τ) be a topological space. If every proper open subspace of X is monotonically Hausdorff, then X is monotonically Hausdorff.*

Proof. Pick y_1, y_2 and y_3 any three distinct points in X, and let $Y_i = X \backslash \{y_i\}$ for each $i = 1, 2, 3$. Since the subspace Y_i is a monotonically Hausdorff subspace for each $i = 1, 2, 3$, there exists a monotone Hausdorff operator $g_i : Y_i \times Y_i \to \tau_{Y_i}$.

Define a monotone Hausdorff operator $g : (X \times X) \backslash \Delta \to \tau$ by

$$g(x, y) = \left\{ \begin{array}{ll} g_1(x, y) & if\ x \neq y_1 \neq y \\ g_2(x, y_1) & if\ x \neq y_2\ and\ y = y_1 \\ g_2(y_1, y) & if\ x = y_1\ and\ y \neq y_2 \\ g_3(x, y) & if\ \{x, y\} = \{y_1, y_2\} \end{array} \right] .$$

So, $g(x, y) \cap g(y, x) = \phi$ for every pair of distinct points x, y in X. For the condition (2), let M be a subset of X, and assume that $x \notin \overline{M}$. So, we have the following cases for x.

(**I**) If $x = y_1$, so, we have two possibilities: (a) $y_2 \notin M$; (b) $y_2 \in M$.

(I. a) If $y_2 \notin M$, then $g(y, y_1) = g_2(y, y_1)$ for all $y \in M$. Since g_2 is a monotone Hausdorff operator,

$$y_1 \notin \overline{\cup\{g(y, y_1) : y \in M\}}.$$

(I. b) If $y_2 \in M$,

$$\overline{\cup\{g(y, y_1) : y \in M\}} = \overline{(\cup\{g_2(y, y_1) : y \in M \backslash \{y_2\}\})} \cup \overline{g_3(y_2, y_1)},$$

and since g_2, g_3 are monotone Hausdorff operators, $y_1 \notin \overline{\cup\{g(y, y_1) : y \in M\}}$.

(**II**) If $x \neq y_1$, then we have four possibilities: (a) $y_1, y_2 \in M$; (b) $y_1 \notin M$, $y_2 \in M$; (c) $y_1 \notin M$, $y_2 \notin M$; (d) $y_1 \in M$, $y_2 \notin M$.

(II. a) In this case,

$$\overline{\cup\{g(y, x) : y \in M\}} = \overline{(\cup\{g_1(y, x) : y \in M \backslash \{y_1\}\})} \cup \overline{g_2(y_1, x)},$$

so, $x \notin \overline{\cup\{g(y, x) : y \in M\}}$.

In the cases (II. b) and (II. c) we have, $g(y, x) = g_1(y, x)$ for all $y \in M$. Hence, $x \notin \overline{\cup\{g(y, x) : y \in M\}}$.

(II. d) We have,

$$g(y, x) = \begin{cases} g_3(y_1, y_2) & if\ x = y_2 \text{ and } y = y_1 \\ g_2(y_1, x) & if\ x \neq y_2 \text{ and } y = y_1 \\ g_1(y, y_2) & if\ x = y_2 \text{ and } y \neq y_1 \\ g_1(y, x) & if\ x \neq y_2 \text{ and } y \neq y_1 \end{cases} .$$

Thus, $x \notin \overline{\cup\{g(y, x) : y \in M\}}$. Therefore, X is a monotonically Hausdorff space. ■

Corollary 3.2 *Monotonically Hausdorff is a strongly properly open hereditary property.* ■

In fact, we do not know whether the monotone Hausdorff operator g in the proof of Theorem 3.1 is a strong (resp., acyclic) monotone Hausdorff operator or not, even if the operators g_i are strong (resp., acyclic). Thus, we have the following questions.

Question 4. *Is strongly monotonically Hausdorff a properly hereditary property?*

Question 5. *Is acyclically monotonically Hausdorff a properly hereditary property?*

3.2 Some Charachterizations of Monotonically T_2 Spaces

In the following, we shall show that the condition (1) in Definition 2.1 is not necessary. Moreover, we shall construct a monotone Hausdorff operator h from any monotone Hausdorff operator g, such that the open neighborhoods $h(x, y)$, $h(y, x)$ have disjoint closures for all distinct points x, y, also, we shall give some characterizations of monotone Hausdorff spaces.

Theorem 3.3 *A space (X, τ) is a monotonically Hausdorff space iff there is a function $k : (X \times X)\backslash\Delta \to \tau$ assigning to each ordered pair (x, y) of distinct points in X an open neighborhood $k(x, y)$ of only one point of the set $\{x, y\}$, and satisfying the condition:*

For each $M \subseteq X$, if $x \in \overline{\cup\{k(y, x) : y \in M\}}$, then $x \in \overline{M}$. (6)

Proof. For sufficiency, first, we shall show that X is T_1. Suppose that X is not T_1. Therefore there exist two distinct points x_0, y_0 in X, such that we can not separate them by open sets. Without loss of generality, assume that $y_0 \in k(x_0, y_0)$. Now, $y_0 \in k(x_0, y_0) \subseteq \overline{k(x_0, y_0)}$, but $y_0 \notin \overline{\{x_0\}}$, contradiction.

Define the function $h : (X \times X) \backslash \Delta \to \tau$ as follows:

$$h(x, y) = \left\{ \begin{array}{ll} k(y, x) & \textit{if } x \notin k(x, y) \\ k(x, y) & \textit{if } x \in k(x, y) \end{array} \right],$$

for all $x \neq y$ in X. Therefore, $h(x, y)$ is an open neighborhood of x and does not contain y. Also h satisfies the corresponding condition of condition (6), that is,

$$\text{for each } M \subseteq X, \text{ if } x \in \overline{\cup\{h(y, x) : y \in M\}}, \text{ then } x \in \overline{M}.$$

Moreover, if A is a closed set and $x \notin A$, then, $x \notin \overline{\cup\{h(y, x) : y \in A\}}$. Hence, there exists an open neighborhood U_x of x such that

$$U_x \cap (\cup h\{(y, x) : y \in A\}) = \phi.$$

Thus, X is regular.

Now, for each $x \neq y$ in X, there exist two open neighborhoods U_x and U_y of x and y, respectively, with $\overline{U_x} \cap \overline{U_y} = \phi$. Define the function $g : (X \times X) \backslash \Delta \to \tau$ as follows:

$$g(x, y) = h(x, y) \cap U_x.$$

So, $g(x, y) \cap g(y, x) = \phi$.

For the condition (2), assume that $x \in \overline{\cup\{g(y, x) : y \in M\}}$, since $g(y, x) \subseteq h(y, x)$, $x \in \overline{\cup\{h(y, x) : y \in M\}}$, and thus $x \in \overline{M}$. Hence g is a monotone Hausdorff operator, therefore, X is a monotonically Hausdorff space. ■

Remark 1 *For the function k in the statement of Theorem 3.3, it is not possible that x belongs to both sets $k(x, y)$ and $k(y, x)$. In fact, if $x \in k(x, y) \cap k(y, x)$, then $y \notin k(x, y) \cup k(y, x)$, so $k(y, x)$ is an open neighborhood of x and $k(y, x) \cap \{y\} = \phi$, whence $x \notin \overline{\{y\}}$. But the condition (6) implies that $x \in \overline{\{y\}}$, because $x \in k(y, x) \subseteq \overline{k(y, x)}$.*

In the next theorem, we shall strengthen the condition (1) in Definition 2.1.

Theorem 3.4 *A space (X, τ) is a monotonically Hausdorff space iff there is a function $k : (X \times X) \backslash \Delta \to \tau$ assigning to each ordered pair (x, y) of distinct points in X an open neighborhood $k(x, y)$ of x, satisfying the condition (6) and satisfying the following condition:*

$$\overline{k(x, y)} \cap \overline{k(y, x)} = \phi. \tag{7}$$

Proof. Assume that (X, τ) is a monotonically Hausdorff space with a monotone Hausdorff operator $g : (X \times X)\backslash\Delta \to \tau$. Since X is regular, for each pair of distinct points x, y in X there exists an open neighborhood $U_{x,y}$ of x such that

$$x \in U_{x,y} \subseteq \overline{U_{x,y}} \subseteq g(x, y).$$

Now, define the function $k : (X \times X)\backslash\Delta \to \tau$ by

$$k(x, y) = U_{x,y}.$$

Then g satisfies the conditions (6) and (7). In fact, if $x \neq y$ in X, then $\overline{k(x, y)} \subseteq g(x, y)$ and $\overline{k(y, x)} \subseteq g(y, x)$, so $\overline{k(x, y)} \cap \overline{k(y, x)} = \phi$. For the condition (6), assume that $x \in \overline{\cup\{k(y, x) : y \in M\}}$. Since $k(x, y) \subseteq g(x, y)$, we have $x \in \overline{\cup\{g(y, x) : y \in M\}}$, which implies that $x \in \overline{M}$. ■

The characterization in Theorem 3.4 would seem to be a stronger property than the monotonically Hausdorff property. However, it is easy to prove the following result.

Corollary 3.5 *Let (X, τ) be a topological space. The following are equivalent:*

a) the space (X, τ) is a monotonically Hausdorff space.

b) there is a function $k : (X \times X)\backslash\Delta \to \tau$ assigning to each ordered pair (x, y) of distinct points in X an open neighborhood $k(x, y)$ of x, satisfying the condition (7) and satisfying the following condition:

$$\text{For each } M \subseteq X, \text{ if } x \in \overline{\cup\{\overline{k(y, x)} : y \in M\}}, \text{ then } x \in \overline{M}. \qquad (8)$$

c) there is a function $h : (X \times X)\backslash\Delta \to \tau$ assigning to each ordered pair (x, y) of distinct points in X an open neighborhood $h(x, y)$ of only one point of the set $\{x, y\}$, and satisfying the condition (8). ■

Now, it is obvious that condition (1) in Definition 2.1 is redundant. Thus, we have the following:

Corollary 3.6 *A space (X, τ) is a strongly monotonically Hausdorff space iff there is a function $g : (X \times X)\backslash\Delta \to \tau$ assigning to each ordered pair (x, y) of distinct points in X an open neighborhood $g(x, y)$ of only one point of the set $\{x, y\}$, and satisfying the conditions (2) and (3).* ■

3.3 Hausdorff Spaces

Return back to condition (6) in Theorem 3.3, and weaken it in the following sense: *For each $M \subseteq X$, if $x \in \cup\{\overline{g(y,x)} : y \in M\}$*, then $x \in \overline{M}$. This would seem to be a weaker monotonic property than monotonically Hausdorff property. Actually, this is not the case, but it will gives us the following result.

Theorem 3.7 *A topological space $(X,\ \tau)$ is Hausdorff iff there exists a function $g : (X \times X)\backslash\Delta \to \tau$ which assigns to each ordered pair $(x,\ y)$ of distinct points in X an open neighborhood $g(x,\ y)$ of x which does not contain y and satisfies the following condition:*

$$\text{For each } M \subseteq X, \text{ if } x \in \cup\{\overline{g(y,\ x)} : y \in M\}, \text{ then } x \in \overline{M}. \tag{9}$$

Proof. For each $x \neq y$ in X, there exist two disjoint open neighborhoods U_x and U_y of x and y, respectively. Then, the set $U_x \cap (X\backslash\overline{U_y})$ is an open neighborhood of x. Define the function $g : (X \times X)\backslash\Delta \to \tau$ by:

$$g(x,\ y) = U_x \cap (X\backslash\overline{U_y})$$

Now, let $\overline{M} \subseteq X\backslash\{x\}$, hence $x \notin \overline{g(y,\ x)}$ for all $y \in M$, because;

$$\overline{g(y,\ x)} = \overline{U_y \cap (X\backslash\overline{U_x})} \subseteq \overline{U_y}.$$

Thus, $x \notin \overline{M}$ implies that $x \notin \cup\{\overline{g(y,\ x)} : y \in M\}$.

Conversely, assume that the function g exists. Let $x \neq y$ in X, so, $x \in g(x,\ y)$ and $y \notin g(x,\ y)$. Then X is T_1. Since $x \notin \{y\} = \overline{\{y\}} \subseteq \overline{g(y,\ x)}$, so, define the function $h : (X \times X)\backslash\Delta \to \tau$ by

$$h(x,\ y) = g(x,\ y) \cap (X\backslash\overline{g(y,\ x)}).$$

Now, $h(x,\ y)$ and $h(y,\ x)$ are disjoint open neighborhoods of x and y, respectively. ■

Recall that a space $(X,\ \tau)$ is a Urysohn space iff for every pair of distinct points $x,\ y$ in X there exist two open neighborhoods $U,\ V$ such that $x \in U$, $y \in V$ and $\overline{U} \cap \overline{V} = \phi$.

Corollary 3.8 *A topological space (X, τ) is a Urysohn space iff there exists a function $k : (X \times X)\backslash\Delta \to \tau$ which assigns to each ordered pair $(x,\ y)$ of distinct points in X an open neighborhood $k(x,\ y)$ of x, such that the conditions (7) and (9) are satisfying.* ■

We do not know what happens if condition (3) in Definition 2.1 is combined with the statement in Theorem 3.7, so we ask the following question:

Question 6. *Assume that there exists a function* $g : (X \times X)\backslash\Delta \to \tau$ *which assigns to each ordered pair* (x, y) *of distinct points in* X *an open neighborhood* $g(x, y)$ *of* x *that does not contain* y *and that satisfies the conditions (3) and (9). Are these conditions equivalent to Hausdorffness? Monotonically Hausdorffness? Strongly monotonically Hausdorffness? Does this gives us a new class of spaces?*

Since strongly monotonically Hausdorff implies monotonically Hausdorff, thus, if the answer of question 6 is positive for any part, then the answer will be negative for the other parts, provided that, the answer of Question 1 is negative.

References

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