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Monotonic Solutions of a Class of Quadratic Singular Integral Equations of Volterra type

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Abstract

We study nonlinear singular integral equation of Volterra type in Banach space of real functions defined and continuous on a bounded and closed interval. Using a suitable measure of noncompactness we prove the existence of monotonic solutions. Also a generalized result is taken in the consideration.

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1 Preliminaries and Introduction

In this paper, we are going to study the solvability of a nonlinear singular integral equation of Volterra type of the form:

$$x(t) = a(t) + x(t) \int_0^t \frac{v(t, s, x(s))}{(t-s)^{\alpha}} ds, \tag{1}$$

where $t \in I = [0, M]$, $M < \infty$ *and* $0 < \alpha \leq 1$.

We look for solutions of that equation in the Banach space of real functions being defined and continuous on a bounded and closed interval. The main tool used in our investigations is a special measure of noncompactness constructed in such a way enable us to study the solvability of considered equations in the class of monotonic functions.

For further purposes, we collect a few auxiliary results which will be needed in the sequel.

Assume that $(E, \|.\|)$ is an infinite-dimensional Banach space with the zero element θ. Denote by $B(x, r)$ the closed ball centered at x and with radius r. The symbol B_r stands for the ball $B(\theta, r)$.

If X is a subset of E , then $\overline{X}$, Conv X denote the closure and convex closure of X , respectively.

We use the symbols λX and $X + Y$ to denote the algebraic operations on sets. The family of all nonempty and bounded subsets of E will be denoted by $\mathfrak{M}_E$ and its subfamily consisting of all relatively compact sets is denoted by $\mathfrak{N}_E$.

Throughout this section, we accept the following definition of the notion of a measure of noncompactness.

Definition 1.1 *A function $\mu : \mathfrak{M}_E \longrightarrow \mathbb{R}_+ = [0, \infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:*

1^0 *the family* $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ *is nonempty and* $\ker \mu \subset \mathfrak{N}_E$;

2^0 $X \subset Y \Longrightarrow \mu(X) \leq \mu(Y)$;

3^0 $\mu(\overline{X}) = \mu(Conv\, X) = \mu(X)$;

4^0 $\mu(\lambda X + (1-\lambda) Y) \leq \lambda \mu(X) + (1-\lambda) \mu(Y)$ *,for* $x \in [0, 1]$;

5^0 *if* (X_n) *is a sequence of closed sets from* $\mathfrak{M}_E$ *such that* $X_{n+1} \subset X_n$*,for* $n = 1, 2, \ldots,$

and if $\lim_{n \longrightarrow \infty} \mu(X_n) = 0$*,then the set* $X_\infty = \bigcap_{n=1}^{\infty} X_n$ *is nonempty.*

The family $\ker\mu$ described in 1^0 is called the kernel of the measure of noncompactness μ.

Further facts concerning measures of noncompactness and its properties may be found in [4].

For our further purposes, we shall only need the following fixed-point theorem [8].

Theorem 1.2 *Let Q be a nonempty bounded closed convex subset of the space E and let $F : Q \to Q$ be a continuous transformation such that $\mu(FX) \leq K\mu(X)$ for any nonempty subset X of Q, where $K \in [0, 1)$ is a constant. Then F has a fixed point in the set Q.*

Remark 1.3 *Under the assumptions of the above theorem, it can be shown that the set $FixF$ of fixed points of F belonging to Q is a member the family* $\ker\mu$ *. This fact permits us to characterize solutions of considered operator equations.*

In what follows, we shall work in the classical Banach space $C[0, M]$ consisting of all real functions defined and continuous on the interval $[0, M]$. For convenience, we write $I = [0, M]$ and $C(I) = C[0, M]$. The space $C(I)$ is furnished by the standard norm $\|x\| = \max\{|x(t)| : t \in I\}$.

Now, we recall the definition of a measure of noncompactness in $C(I)$ which will be used in our further investigations. That measure was introduced and studied in [4].

To do this, let us fix a nonempty and bounded subset X of $C(I)$. For $x \in X$ and $\varepsilon \geq 0$ denoted by $\omega(x, \varepsilon)$, *the modulus of continuity* of the function x, i.e.,

$$\omega(x, \varepsilon) = \sup\{|x(t) - x(\tau)| : t, \tau \in I, |t - \tau| \leq \varepsilon\}.$$

Further, let us put

$$\begin{aligned} \omega(X, \varepsilon) &= \sup\{\omega(x, \varepsilon) : x \in X\}, \\ \omega_o(X) &= \lim_{\varepsilon\to 0}\omega(X, \varepsilon). \end{aligned}$$

Next, let us define the following quantities:

$$\begin{aligned} d(x) &= \sup\{|x(\tau) - x(t)| - [x(\tau) - x(t)] : t, \tau \in I, t \leq \tau\}, \\ i(x) &= \sup\{|x(t) - x(\tau)| - [x(t) - x(\tau)] : t, \tau \in I, t \leq \tau\}, \\ d(X) &= \sup\{d(x) : x \in X\}, \\ i(X) &= \sup\{i(x) : x \in X\}. \end{aligned}$$

Observe that $d(X) = 0$ if and only if all functions belonging to X are nondecreasing on I. In a similar way, we can characterize the set X with $i(X) = 0$. Finally, we define the function μ on the family $\mathfrak{M}_{C(I)}$ by putting

$$\mu(X) = \omega_o(X) + d(X).$$

It can be shown (see [4]) that the function μ is a measure of noncompactness in the space $C(I)$.

The kernel $\ker\mu$ of this measure contains nonempty and bounded sets X such that functions from X are equicontinuous and nondecreasing on the interval I.

Remark 1.4 *The above described properties of the kernel* $\ker\mu$ *of the measure of noncompactness* μ *in conjunction with Remark (1.3) allow us to characterize solutions of the nonlinear integtal equation considered in the next section.*

Remark 1.5 *Observe that, in a similar way, we can define the measure of noncompactness associated with the set quantity* $i(X)$ *defined above. We omit the details concerning that measure.*

2 Main Results

In this section, we shall study the solvability of nonlinear quadratic singular integral equation of Volterra type(1).
We shall look for solutions of that equation in the Banach space of real functions being defined and continuous on a bounded and closed interval.
The tool used in our investigations is a special measure of noncompactness constructed in such a way that its use enables us to study the solvability of considered equation in the class of monotonic fuctions.

First, in equation(1)we notice that the functions $a = a(t)$ and $v = v(t, s, x)$ are given while $x = x(t)$ is unknown function.

We shall investigate equation (1) assuming that the following set of hypotheses is satisfied:

(i) $a \in C(I)$ and the function a is nondecreasing and nonnegative on I;

(ii) $v : I \times I \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $v : I \times I \times \mathbb{R}_+ \to \mathbb{R}_+$ and for arbitrary fixed $s \in I$ and $x \in \mathbb{R}_+$ the function $t \to v(t, s, x)$ is nondecreasing on I;

(iii) there exists a nondecreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that the inequality

$$|v(t, s, x)| \leq f(|x|)$$

holds for all $t, s \in I$ and $x \in \mathbb{R}$;
(iv) the inequality

$$\|a\| + r\frac{M^{1-\alpha}}{1-\alpha} f(r) \leq r$$

has a positive solution r_0 such that $\frac{M^{1-\alpha}}{1-\alpha} f(r_o) \leq 1$. Now, we can formulate our main existence result.

Theorem 2.1 *Under assumptions* $(i) - (iv)$, *equation(1)has at least one solution* $x = x(t)$ *which belonging to the space* $C(I)$ *and is nondecreasing on the interval* I.

Proof. Let us consider the operator A defined on the space $C(I)$ in the following way:

$$(Ax)(t) = a(t) + x(t) \int_0^t \frac{v(t, s, x(s))}{(t-s)^\alpha} ds.$$

In view of assumptions (i) and (ii), it follows that the function Ax is continuous on I for any function $x \in C(I)$, i.e., A transforms the space $C(I)$ into itself.
Moreover, keeping in mind assumptions (iii), we get:

$$\begin{aligned}
|(Ax)(t)| &\leq |a(t)| + |x(t)| \left| \int_0^t \frac{v(t, s, x(s))}{(t-s)^\alpha} ds \right| \\
&\leq \|a\| + \|x\| \int_0^t \frac{f(|x|)}{(t-s)^\alpha} ds \\
&\leq \|a\| + \|x\| f(\|x\|) \int_0^t (t-s)^{-\alpha} ds \\
&\leq \|a\| + \|x\| \frac{t^{1-\alpha}}{1-\alpha} f(\|x\|) \\
&\leq \|a\| + \|x\| \frac{M^{1-\alpha}}{1-\alpha} f(\|x\|).
\end{aligned}$$

Hence,

$$\|Ax\| \leq \|a\| + \|x\| \frac{M^{1-\alpha}}{1-\alpha} f(\|x\|).$$

Thus, taking into account assumption (iv), we infer that there exists $r_0 > 0$ with $\frac{M^{1-\alpha}}{1-\alpha} f(r_o) \leq 1$ and such that the operator A transforms the ball B_{r_o}into itself.

In what follows, we shall consider the operator A on the subset $B_{r_o}^{+}$ of the ball B_{r_o} defined in the following way:

$$B_{r_o}^{+} = \{x \in B_{r_o} : x(t) \geq 0, for t \in I\}.$$

Obviously, the set $B_{r_o}^{+}$ is nonempty , bounded , closed and convex. In view of these facts and assumptions (i) and (ii), we deduce that A transforms the set $B_{r_o}^{+}$into itself.
Now, we shall show that A is contiguous on the set $B_{r_o}^{+}$. To do this, let us fix $\varepsilon > 0$ and take arbitrary $x, y \in B_{r_o}^{+}$ such that $\|x - y\| \leq \varepsilon$.
Then, for $t \in I$, we derive the following estimates:

$$\begin{aligned}
|(Ax)(t) - (Ay)(t)| &\leq \left| x(t) \int_0^t \frac{v(t,s,x(s))}{(t-s)^{\alpha}} ds - y(t) \int_0^t \frac{v(t,s,y(s))}{(t-s)^{\alpha}} ds \right| \\
&\leq \left| x(t) \int_0^t \frac{v(t,s,x(s))}{(t-s)^{\alpha}} ds - y(t) \int_0^t \frac{v(t,s,x(s))}{(t-s)^{\alpha}} ds \right| \\
&+ \left| y(t) \int_0^t \frac{v(t,s,x(s))}{(t-s)^{\alpha}} ds - y(t) \int_0^t \frac{v(t,s,y(s))}{(t-s)^{\alpha}} ds \right| \\
&\leq \|x - y\| \int_0^t \frac{|v(t,s,x(s))|}{(t-s)^{\alpha}} ds + \|y\| \int_0^t \frac{|v(t,s,x(s)) - v(t,s,y(s))|}{(t-s)^{\alpha}} ds \\
&\leq \varepsilon f(r_o) \int_0^t (t-s)^{-\alpha} ds + r_o \beta_{r_o}(\varepsilon) \int_0^t (t-s)^{-\alpha} ds \\
&\leq \varepsilon f(r_o) \frac{M^{1-\alpha}}{1-\alpha} + r_o \beta_{r_o}(\varepsilon) \frac{M^{1-\alpha}}{1-\alpha},
\end{aligned}$$

where we denoted

$$\beta_{r_o}(\varepsilon) = \sup\{|v(t,s,x) - v(t,s,y)| : t, s \in I, x, y \in [0, r_o], |x - y| \leq \varepsilon\}.$$

Obviously, $\beta_{r_o}(\varepsilon) \to 0$ as $\varepsilon \to 0$ which is a simple consequence of the uniform continuity of the function v on the set $I \times I \times [0, r_o]$.

From the above estimate, we derive the following inequality:

$$\|Ax - Ay\| \leq \varepsilon f(r_o) \frac{M^{1-\alpha}}{1-\alpha} + r_o \beta_{r_o}(\varepsilon) \frac{M^{1-\alpha}}{1-\alpha},$$

which implies the continuity of the operator A on the set B_{ro}^{+}.
In what follows, let us take a nonempty set $X \subset B_{r_o}^{+}$. Further, fix arbitrary number $\varepsilon > 0$ and choose $x \in X$ and $t, \tau \in [0, M]$ such that $|t - \tau| \leq \varepsilon$. Without loss of generality, we may assume that $t \leq \tau$.
Then, in view of our assumptions we obtain

$$|(Ax)(\tau) - (Ax)(t)| \leq |a(\tau) - a(t)| + \left| x(\tau) \int_0^{\tau} \frac{v(\tau,s,x(s))}{(\tau-s)^{\alpha}} ds - x(t) \int_0^t \frac{v(t,s,x(s))}{(t-s)^{\alpha}} ds \right|$$

$$\begin{aligned}
&\leq w(a,\varepsilon)+\left|x(\tau)\int_0^\tau \frac{v(\tau,s,x(s))}{(\tau-s)^\alpha}ds - x(t)\int_0^\tau \frac{v(\tau,s,x(s))}{(\tau-s)^\alpha}ds\right| \\
&+ \left|x(t)\int_0^\tau \frac{v(\tau,s,x(s))}{(\tau-s)^\alpha}ds - x(t)\int_0^\tau \frac{v(t,s,x(s))}{(\tau-s)^\alpha}ds\right| \\
&+ \left|x(t)\int_0^\tau \frac{v(t,s,x(s))}{(\tau-s)^\alpha}ds - x(t)\int_0^\tau \frac{v(t,s,x(s))}{(t-s)^\alpha}ds\right| \\
&+ \left|x(t)\int_0^\tau \frac{v(t,s,x(s))}{(t-s)^\alpha}ds - x(t)\int_0^t \frac{v(t,s,x(s))}{(t-s)^\alpha}ds\right| \\
&\leq \omega(a,\varepsilon)+|x(\tau)-x(t)|\int_0^\tau \frac{|v(\tau,s,x(s))|}{(\tau-s)^\alpha}ds \\
&+ |x(t)|\int_0^\tau \frac{|v(\tau,s,x(s))-v(t,s,x(s))|}{(\tau-s)^\alpha}ds \\
&+ |x(t)|\int_0^\tau |v(t,s,x(s))|\left[\frac{1}{(\tau-s)^\alpha}-\frac{1}{(t-s)^\alpha}\right]ds \\
&+ |x(t)|\int_t^\tau |v(t,s,x(s))|\left[\frac{1}{(t-s)^\alpha}\right]ds \\
&\leq \omega(a,\varepsilon)+\omega(x,\varepsilon)\int_0^\tau f(r_o)\frac{1}{(\tau-s)^\alpha}ds + r_o\int_0^\tau \gamma_{r_o}(\varepsilon)\frac{1}{(\tau-s)^\alpha}ds \\
&+ r_o\int_0^\tau f(r_o)\left[\frac{1}{(\tau-s)^\alpha}-\frac{1}{(t-s)^\alpha}\right]ds + r_o\int_t^\tau f(r_o)\left[\frac{1}{(t-s)^\alpha}\right]ds \\
&\leq \omega(a,\varepsilon)+\omega(x,\varepsilon)f(r_o)\frac{\tau^{1-\alpha}}{1-\alpha}+r_o\gamma_{r_o}(\varepsilon)\frac{\tau^{1-\alpha}}{1-\alpha} \\
&+ r_of(r_o)\left[\frac{\tau^{1-\alpha}}{1-\alpha}+\frac{(t-\tau)^{1-\alpha}}{1-\alpha}-\frac{t^{1-\alpha}}{1-\alpha}\right]+r_of(r_o)\left[-\frac{(t-\tau)^{1-\alpha}}{1-\alpha}\right] \\
&\leq \omega(a,\varepsilon)+\omega(x,\varepsilon)f(r_o)\frac{M^{1-\alpha}}{1-\alpha}+r_o\gamma_{r_o}(\varepsilon)\frac{M^{1-\alpha}}{1-\alpha} \\
&+ r_of(r_o)\left[\frac{\tau^{1-\alpha}}{1-\alpha}-\frac{t^{1-\alpha}}{1-\alpha}\right],
\end{aligned}$$

where we have denoted

$$\gamma_{r_o}(\varepsilon)=\sup\{|v(\tau,s,x)-v(t,s,x)| : t,\tau\in I, |\tau-t|\leq\varepsilon, x\in[0,r_o]\}.$$

By applying *The Mean Value Theorem* on the bracket $\left[\frac{\tau^{1-\alpha}}{1-\alpha}-\frac{t^{1-\alpha}}{1-\alpha}\right]$, we get

$$\left[\frac{\tau^{1-\alpha}}{1-\alpha}-\frac{t^{1-\alpha}}{1-\alpha}\right]=\frac{(\tau-t)}{\delta^\alpha}<\frac{\varepsilon}{\delta^\alpha},$$

for all $t<\delta<\tau$.

Then we get

$$|(Ax)(\tau)-(Ax)(t)|\leq\omega(a,\varepsilon)+\omega(x,\varepsilon)f(r_o)\frac{M^{1-\alpha}}{1-\alpha}+r_o\gamma_{r_o}(\varepsilon)\frac{M^{1-\alpha}}{1-\alpha}+r_of(r_o)\frac{\varepsilon}{\delta^\alpha}. \tag{2}$$

Notice that, in view of the uniform continuity of the function v on the set $I \times I \times [0, r_o]$, we have $\gamma_{r_o}(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Now, fix arbitrary $x \in X$ and $t, \tau \in I$ such that $t \leq \tau$. Then we have the following chain of estimates:

$$\begin{aligned}
&|(Ax)(\tau) - (Ax)(t)| - [(Ax)(\tau) - (Ax)(t)] \\
= &\left| a(\tau) + x(\tau) \int_0^\tau \frac{v(\tau, s, x(s))}{(\tau - s)^\alpha} ds - a(t) - x(t) \int_0^t \frac{v(t, s, x(s))}{(t - s)^\alpha} ds \right| \\
- &\left[a(\tau) + x(\tau) \int_0^\tau \frac{v(\tau, s, x(s))}{(\tau - s)^\alpha} ds - a(t) - x(t) \int_0^t \frac{v(t, s, x(s))}{(t - s)^\alpha} ds \right] \\
\leq &\{|a(\tau) - a(t)| - [a(\tau) - a(t)]\} + \left| x(\tau) \int_0^\tau \frac{v(\tau, s, x(s))}{(\tau - s)^\alpha} ds - x(t) \int_0^t \frac{v(t, s, x(s))}{(t - s)^\alpha} ds \right| \\
- &\left[x(\tau) \int_0^\tau \frac{v(\tau, s, x(s))}{(\tau - s)^\alpha} ds - x(t) \int_0^t \frac{v(t, s, x(s))}{(t - s)^\alpha} ds \right],
\end{aligned}$$

and since $a(t)$ is nondecreasing, we can deduce, according to the definition of $d(x)$, that:

$$d(C(I)) = \sup\{d(a) : a \in C(I)\} = 0.$$

Then,

$$\begin{aligned}
&|(Ax)(\tau) - (Ax)(t)| - [(Ax)(\tau) - (Ax)(t)] \\
\leq &\left| x(\tau) \int_0^\tau \frac{v(\tau, s, x(s))}{(\tau - s)^\alpha} ds - x(t) \int_0^t \frac{v(t, s, x(s))}{(t - s)^\alpha} ds \right| \\
- &\left[x(\tau) \int_0^\tau \frac{v(\tau, s, x(s))}{(\tau - s)^\alpha} ds - x(t) \int_0^t \frac{v(t, s, x(s))}{(t - s)^\alpha} ds \right] \\
\leq &\left| x(\tau) \int_0^\tau \frac{v(\tau, s, x(s))}{(\tau - s)^\alpha} ds - x(t) \int_0^\tau \frac{v(\tau, s, x(s))}{(\tau - s)^\alpha} ds \right| \\
+ &\left| x(t) \int_0^\tau \frac{v(\tau, s, x(s))}{(\tau - s)^\alpha} ds - x(t) \int_0^t \frac{v(t, s, x(s))}{(t - s)^\alpha} ds \right| \\
- &\left[x(\tau) \int_0^\tau \frac{v(\tau, s, x(s))}{(\tau - s)^\alpha} ds - x(t) \int_0^\tau \frac{v(\tau, s, x(s))}{(\tau - s)^\alpha} ds \right] \\
- &\left[x(t) \int_0^\tau \frac{v(\tau, s, x(s))}{(\tau - s)^\alpha} ds - x(t) \int_0^t \frac{v(t, s, x(s))}{(t - s)^\alpha} ds \right] \\
\leq &|x(\tau) - x(t)| \int_0^\tau \frac{v(\tau, s, x(s))}{(\tau - s)^\alpha} ds + |x(t)| \left| \int_0^\tau \frac{v(\tau, s, x(s))}{(\tau - s)^\alpha} ds - \int_0^t \frac{v(t, s, x(s))}{(t - s)^\alpha} ds \right| \\
- &[x(\tau) - x(t)] \int_0^\tau \frac{v(\tau, s, x(s))}{(\tau - s)^\alpha} ds - x(t) \left[\int_0^\tau \frac{v(\tau, s, x(s))}{(\tau - s)^\alpha} ds - \int_0^t \frac{v(t, s, x(s))}{(t - s)^\alpha} ds \right] \\
\leq &\{|x(\tau) - x(t)| - [x(\tau) - x(t)]\} \int_0^\tau \frac{v(\tau, s, x(s))}{(\tau - s)^\alpha} ds \\
+ &x(t) \left\{ \int_0^\tau v(\tau, s, x(s)) \left| (\tau - s)^{-\alpha} - (t - s)^{-\alpha} \right| ds \right.
\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^\tau |v(\tau,s,x(s)) - v(t,s,x(s))|(t-s)^{-\alpha}ds + \int_t^\tau \frac{v(t,s,x(s))}{(t-s)^\alpha}\Bigg\} ds \\
&- x(t)\left\{\int_0^\tau v(\tau,s,x(s))\left[(\tau-s)^{-\alpha} - (t-s)^{-\alpha}\right]ds\right. \\
&+ \int_0^\tau [v(\tau,s,x(s)) - v(t,s,x(s))](t-s)^{-\alpha}ds + \int_t^\tau \frac{v(t,s,x(s))}{(t-s)^\alpha}\Bigg\} ds.
\end{aligned}
$$

Again,as above, let us applying *The Mean Value Theorem* on the bracket $[(\tau-s)^{-\alpha} - (t-s)^{-\alpha}]$, we get

$$
\left[(\tau-s)^{-\alpha} - (t-s)^{-\alpha}\right] = -\alpha\frac{(\tau-t)}{\delta_1^{1+\alpha}} < -\alpha\frac{\varepsilon}{\delta_1^{1+\alpha}},
$$

for all $t < \delta_1 < \tau$. Then we obtain the following inequality:

$$
\begin{aligned}
&|(Ax)(\tau) - (Ax)(t)| - [(Ax)(\tau) - (Ax)(t)] \\
&\le \{|x(\tau) - x(t)| - [x(\tau) - x(t)]\}\int_0^\tau \frac{f(r_o)}{(\tau-s)^\alpha}ds \\
&+ 2\alpha\frac{\varepsilon}{\delta_1^{1+\alpha}}x(t)\int_0^\tau v(\tau,s,x(s))ds \\
&+ x(t)\int_0^\tau \{|v(\tau,s,x(s)) - v(t,s,x(s))| - [v(\tau,s,x(s)) - v(t,s,x(s))]\}(t-s)^{-\alpha}ds.
\end{aligned}
$$

Taking into account that the last term in the above inequality will be vanished (notice that the function $t \to v(t,s,x)$ is nondecreasing on I).
Finally we get

$$
|(Ax)(\tau) - (Ax)(t)| - [(Ax)(\tau) - (Ax)(t)]
$$

$$
\le \{|x(\tau) - x(t)| - [x(\tau) - x(t)]\}\frac{M^{1-\alpha}}{1-\alpha}f(r_o) + 2\alpha\frac{\varepsilon}{\delta_1^{1+\alpha}}x(t)\int_0^\tau v(\tau,s,x(s))ds. \tag{3}
$$

By adding Eq.(2) and Eq.(3) and taking the supremum of the resultant inequality then let $\varepsilon \to 0$, keeping in mind the definition of the measure of noncompactness $\mu(X) = \omega_o(X) + d(X)$, therefore we obtain

$$
\mu(AX) \le \frac{M^{1-\alpha}}{1-\alpha}f(r_o)\mu(X).
$$

Now, taking into account the above inequality and the fact that $\frac{M^{1-\alpha}}{1-\alpha}f(r_o) < 1$ and applying Theorem(1.2), we complete the proof.

Remark 2.2 *Taking into account Remarks (1.3) and (1.4) and the description of the kernel of the measure of noncompactness* μ *given in section 1, we deduce easily from the proof of Theorem (2.1) that the solutions of the integral equation (1) belonging to the set* $B_{r_o}^+$ *are nondecreasing and continuous on the interval* I.

Moreover, those solutions are also positive provided $a(t) > 0$ for $t \in I$.

3 Generalized Results

The results in this section generalize and complete the results in section (2).

We consider the following nonlinear singular integral equation of Volterra type:

$$x(t) = a(t) + (Bx)(t) \int_0^t \frac{v(t, s, x(s))}{(t-s)^\alpha} ds, \tag{4}$$

where the operator B satisfies the following set of conditions:

(i') The operator $B : C(I) \to C(I)$ is continuous and satisfies the conditions of Theorem (1.2) for the measure of noncompactness μ with a constant K and, moreover, B is a positive operator, i.e., $Bx \geq 0$ if $x \geq 0$.

(ii') There exist nonnegative constants b and c such that:

$$|(Bx)(t)| \leq b + c\|x\|,$$

for each $x \in C(I)$ and $t \in I$.

We replace the assumption (iv) in Section (2) with the following assumption:

(iii') The inequality

$$\|a\| + (b + cr) \frac{M^{1-\alpha}}{1-\alpha} f(r) \leq r$$

has a positive solution r_o such that $K \frac{M^{1-\alpha}}{1-\alpha} f(r_o) < 1$.

By connection between the assumptions $(i) - (iii)$, in Section (2), and the assumptions $(i') - (iii')$ we can formulate the following existence result.

Theorem 3.1 *Under assumptions $(i) - (iii)$ and $(i') - (iii')$, the equation (4) has at least one solution $x = x(t)$ which belongs to the space $C(I)$ and is nondecreasing on the interval I.*

Proof. Let us consider the operator V defined on the space $C(I)$ in the following way:

$$(Vx)(t) = a(t) + (Bx)(t) \int_0^t \frac{v(t, s, x(s))}{(t-s)^\alpha} ds.$$

In a similar way as in Proof of Theorem (2.1), we get the following estimates:

1. $|(Vx)(t)| \leq \|a\| + (b + c\|x\|) \dfrac{M^{1-\alpha}}{1-\alpha} f(\|x\|),$

which proves that V transforms the space $C(I)$ into itself.

2. $|(Vx)(t)-(Vy)(t)| \leq \|Bx-By\| f(r_o)\dfrac{M^{1-\alpha}}{1-\alpha}+(b+cr_o)\beta_{r_o}(\varepsilon)\dfrac{M^{1-\alpha}}{1-\alpha},$

and from the uniform continuity of the function v on the set $I\times I\times[0,r_o]$ and the continuity of V, the last inequality implies the continuity of the operator V on the set $B^+_{r_o}$.

3. $|(Vx)(\tau)-(Vy)(t)|$

$$
\begin{aligned}
\leq\ & |a(\tau)-a(t)|+\left|(Bx)(\tau)\int_0^\tau \frac{v(\tau,s,x(s))}{(\tau-s)^\alpha}ds-(Bx)(t)\int_0^t \frac{v(t,s,x(s))}{(t-s)^\alpha}ds\right| \\
\leq\ & w(a,\varepsilon)+\left|(Bx)(\tau)\int_0^\tau \frac{v(\tau,s,x(s))}{(\tau-s)^\alpha}ds-(Bx)(t)\int_0^\tau \frac{v(\tau,s,x(s))}{(\tau-s)^\alpha}ds\right| \\
+\ & \left|(Bx)(t)\int_0^\tau \frac{v(\tau,s,x(s))}{(\tau-s)^\alpha}ds-(Bx)(t)\int_0^\tau \frac{v(t,s,x(s))}{(\tau-s)^\alpha}ds\right| \\
+\ & \left|(Bx)(t)\int_0^\tau \frac{v(t,s,x(s))}{(\tau-s)^\alpha}ds-(Bx)(t)\int_0^\tau \frac{v(t,s,x(s))}{(t-s)^\alpha}ds\right| \\
+\ & \left|(Bx)(t)\int_0^\tau \frac{v(t,s,x(s))}{(t-s)^\alpha}ds-(Bx)(t)\int_0^t \frac{v(t,s,x(s))}{(t-s)^\alpha}ds\right| \\
\leq\ & \omega(a,\varepsilon)+|(Bx)(\tau)-(Bx)(t)|\int_0^\tau \frac{|v(\tau,s,x(s))|}{(\tau-s)^\alpha}ds \\
+\ & |(Bx)(t)|\int_0^\tau \frac{|v(\tau,s,x(s))-v(t,s,x(s))|}{(\tau-s)^\alpha}ds \\
+\ & |(Bx)(t)|\int_0^\tau |v(t,s,x(s))|\left[\frac{1}{(\tau-s)^\alpha}-\frac{1}{(t-s)^\alpha}\right]ds \\
+\ & |(Bx)(t)|\int_t^\tau |v(t,s,x(s))|\left[\frac{1}{(t-s)^\alpha}\right]ds \\
\leq\ & \omega(a,\varepsilon)+\omega(Bx,\varepsilon)\int_0^\tau f(r_o)\frac{1}{(\tau-s)^\alpha}ds+(b+cr_o)\int_0^\tau \gamma_{r_o}(\varepsilon)\frac{1}{(\tau-s)^\alpha}ds \\
+\ & (b+cr_o)\int_0^\tau f(r_o)\left[\frac{1}{(\tau-s)^\alpha}-\frac{1}{(t-s)^\alpha}\right]ds+(b+cr_o)\int_t^\tau f(r_o)\left[\frac{1}{(t-s)^\alpha}\right]ds \\
\leq\ & \omega(a,\varepsilon)+\omega(Bx,\varepsilon)f(r_o)\frac{M^{1-\alpha}}{1-\alpha}+(b+cr_o)\gamma_{r_o}(\varepsilon)\frac{M^{1-\alpha}}{1-\alpha}+(b+cr_o)f(r_o)\frac{\varepsilon}{\delta^\alpha}.
\end{aligned}
$$

Hence, we get
$|(Vx)(\tau)-(Vy)(t)|$

$$
\leq \omega(a,\varepsilon)+\omega(Bx,\varepsilon)f(r_o)\frac{M^{1-\alpha}}{1-\alpha}+(b+cr_o)\gamma_{r_o}(\varepsilon)\frac{M^{1-\alpha}}{1-\alpha}+(b+cr_o)f(r_o)\frac{\varepsilon}{\delta^\alpha}. \tag{5}
$$

Notice that, in view of the uniform continuity of the function v on the set $I \times I \times [0, r_o]$, we have $\gamma_{r_o}(\varepsilon) \to 0$ as $\varepsilon \to 0$.

4. $|(Vx)(\tau) - (Vx)(t)| - [(Vx)(\tau) - (Vx)(t)]$

$$
\begin{aligned}
= & \left| a(\tau) + (Bx)(\tau) \int_0^\tau \frac{v(\tau, s, x(s))}{(\tau - s)^\alpha} ds - a(t) - (Bx)(t) \int_0^t \frac{v(t, s, x(s))}{(t - s)^\alpha} ds \right| \\
- & \left[a(\tau) + (Bx)(\tau) \int_0^\tau \frac{v(\tau, s, x(s))}{(\tau - s)^\alpha} ds - a(t) - (Bx)(t) \int_0^t \frac{v(t, s, x(s))}{(t - s)^\alpha} ds \right] \\
\leq & \{ |a(\tau) - a(t)| - [a(\tau) - a(t)] \} + \left| (Bx)(\tau) \int_0^\tau \frac{v(\tau, s, x(s))}{(\tau - s)^\alpha} ds - (Bx)(t) \int_0^t \frac{v(t, s, x(s))}{(t - s)^\alpha} ds \right| \\
- & \left[(Bx)(\tau) \int_0^\tau \frac{v(\tau, s, x(s))}{(\tau - s)^\alpha} ds - (Bx)(t) \int_0^t \frac{v(t, s, x(s))}{(t - s)^\alpha} ds \right] \\
\leq & \left| (Bx)(\tau) \int_0^\tau \frac{v(\tau, s, x(s))}{(\tau - s)^\alpha} ds - (Bx)(t) \int_0^t \frac{v(t, s, x(s))}{(t - s)^\alpha} ds \right| \\
- & \left[(Bx)(\tau) \int_0^\tau \frac{v(\tau, s, x(s))}{(\tau - s)^\alpha} ds - (Bx)(t) \int_0^t \frac{v(t, s, x(s))}{(t - s)^\alpha} ds \right] \\
\leq & \left| (Bx)(\tau) \int_0^\tau \frac{v(\tau, s, x(s))}{(\tau - s)^\alpha} ds - (Bx)(t) \int_0^\tau \frac{v(\tau, s, x(s))}{(\tau - s)^\alpha} ds \right| \\
+ & \left| (Bx)(t) \int_0^\tau \frac{v(\tau, s, x(s))}{(\tau - s)^\alpha} ds - (Bx)(t) \int_0^t \frac{v(t, s, x(s))}{(t - s)^\alpha} ds \right| \\
- & \left[(Bx)(\tau) \int_0^\tau \frac{v(\tau, s, x(s))}{(\tau - s)^\alpha} ds - (Bx)(t) \int_0^\tau \frac{v(\tau, s, x(s))}{(\tau - s)^\alpha} ds \right] \\
- & \left[(Bx)(t) \int_0^\tau \frac{v(\tau, s, x(s))}{(\tau - s)^\alpha} ds - (Bx)(t) \int_0^t \frac{v(t, s, x(s))}{(t - s)^\alpha} ds \right] \\
\leq & \{ |(Bx)(\tau) - (Bx)(t)| - [(Bx)(\tau) - (Bx)(t)] \} \int_0^\tau \frac{v(\tau, s, x(s))}{(\tau - s)^\alpha} ds \\
+ & (Bx)(t) \left\{ \int_0^\tau v(\tau, s, x(s)) \left| (\tau - s)^{-\alpha} - (t - s)^{-\alpha} \right| ds \right. \\
+ & \left. \int_0^\tau |v(\tau, s, x(s)) - v(t, s, x(s))| (t - s)^{-\alpha} ds + \int_t^\tau \frac{v(t, s, x(s))}{(t - s)^\alpha} \right\} ds \\
- & (Bx)(t) \left\{ \int_0^\tau v(\tau, s, x(s)) \left[(\tau - s)^{-\alpha} - (t - s)^{-\alpha} \right] ds \right. \\
+ & \left. \int_0^\tau [v(\tau, s, x(s)) - v(t, s, x(s))] (t - s)^{-\alpha} ds + \int_t^\tau \frac{v(t, s, x(s))}{(t - s)^\alpha} \right\} ds \\
\leq & \{ |(Bx)(\tau) - (Bx)(t)| - [(Bx)(\tau) - (Bx)(t)] \} \int_0^\tau \frac{f(r_o)}{(\tau - s)^\alpha} ds \\
+ & 2\alpha \frac{\varepsilon}{\delta_1^{1+\alpha}} (Bx)(t) \int_0^\tau v(\tau, s, x(s)) ds \\
+ & (Bx)(t) \int_0^\tau \{ |v(\tau, s, x(s)) - v(t, s, x(s))| - [v(\tau, s, x(s)) - v(t, s, x(s))] \} (t - s)^{-\alpha} ds \\
= & \{ |(Bx)(\tau) - (Bx)(t)| - [(Bx)(\tau) - (Bx)(t)] \} \frac{M^{1-\alpha}}{1 - \alpha} f(r_o)
\end{aligned}
$$

$$+ \quad 2\alpha \frac{\varepsilon}{\delta_1^{1+\alpha}}(Bx)(t) \int_0^\tau v\left(\tau, s, x\left(s\right)\right) ds.$$

Hence, we get

$$\begin{aligned} &\left|\left(Vx\right)\left(\tau\right) - \left(Vx\right)\left(t\right)\right| - \left[\left(Vx\right)\left(\tau\right) - \left(Vx\right)\left(t\right)\right] \\ &\qquad \leq \left\{\left|\left(Bx\right)\left(\tau\right) - \left(Bx\right)\left(t\right)\right| - \left[\left(Bx\right)\left(\tau\right) - \left(Bx\right)\left(t\right)\right]\right\} \frac{M^{1-\alpha}}{1-\alpha} f\left(r_o\right) \\ &\qquad\quad + 2\alpha \frac{\varepsilon}{\delta_1^{1+\alpha}}(Bx)(t) \int_0^\tau v\left(\tau, s, x\left(s\right)\right) ds. \qquad (6) \end{aligned}$$

Finally (as in the pervious section), by adding Eq.(5) and Eq.(6)and keeping in mind the definition of the measure of noncompactness μ, we obtain

$$\mu\left(VX\right) \leq \frac{M^{1-\alpha}}{1-\alpha} f\left(r_o\right) \mu\left(BX\right) \leq \frac{M^{1-\alpha}}{1-\alpha} f\left(r_o\right) K \mu\left(X\right).$$

Now, taking into account the above inequality and the fact that $\frac{M^{1-\alpha}}{1-\alpha} f\left(r_o\right) K < 1$ and applying Theorem (1.2),we complete the proof.

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