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A Note on Biquaternionic MIT Bag Model

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Abstract

The term "bag model" unifies several attempts to construct a mathematical description of the phenomenon of quark confinement. Such models are often used for estimating different characteristics of a hadron. The aim of this note is to add one more characterization to the list of known characterizations of Dirac equation (see, e.g., [8], [10]). In the article, we study the Dirichlet type boundary value problems and MIT bag model in three-dimensional piece-wise Liapunov domains.

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1 Introduction

It is well known that the theory of solutions of the Dirac equation reduces, in some degenerate cases, to that of complex holomorphic functions. It is also known that not too many facts from the holomorphic function theory have their extensions onto the Dirac equation theory. In an earlier paper [15], we studied the analogue of the Cauchy-type integral for time-harmonic solutions of the relativistic Dirac equation putting emphasis on the peculiarities of the case of a piece-wise Liapunov surface of integration. The present paper continues the study of Dirac equations begun in [14], [15]. We therefore will use systematically the notations and results from [15]. In the paper we focus on the Dirichlet type boundary value problems and MIT bag model in case of a piece-wise Liapunov surface. In realizing this study we follow the approach presented in references [10], [12] (compare also with reference [8]). The case of smooth (Liapunov) surface has been treated in [10], [12]. It is relevant to add the papers [11] where boundary value problems have been investigated on Lipschitz domains.

2 Time-harmonic bispinor fields theory and the Cauchy-Dirac integral

2.1 Time-harmonic bispinor fields theory

Throughout the paper we will work with the Cauchy-Dirac integral. There are several ways to define the Cauchy-Dirac integral. We will use the approach developed by R. Rocha-Chávez and M. Shapiro (for a detailed exposition see, e.g., [12]).

Let Ω be a domain in $\mathbb{R}^3$, $\Gamma := \partial\Omega$ be its boundary. We consider the following *Dirac equation* for a free massive particle of spin $\frac{1}{2}$:

$$\mathbb{D}[\Phi] := \Big(\gamma_0\partial_t - \sum_{k=1}^{3}\gamma_k\partial_k + im\Big)[\Phi] = 0,$$

where the Dirac matrices have the standard Dirac-Pauli form

$$\gamma_0 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma_1 := \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$
$$\gamma_2 := \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_3 := \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

and where $\partial_t := \frac{\partial}{\partial t}$; $\partial_k := \frac{\partial}{\partial x_k}$, $m \in \mathbb{R}$, $\Phi : \mathbb{R}^4 \to \mathbb{C}^4$. Suppose that the spinor field Φ is time-harmonic (= monochromatic):

$$\Phi(t, x) = q(x)e^{i\omega t}, \tag{2.1}$$

where $\omega \in \mathbb{R}$ is the frequency and $q : \Omega \subset \mathbb{R}^3 \to \mathbb{C}^4$ is the amplitude. Then the relativistic Dirac equation is equivalent to the *time-harmonic Dirac equation*:

$$\mathbb{D}_{\omega,m}[q] := \Big(i\omega\gamma_0 - \sum_{k=1}^{3}\gamma_k\partial_k + im\Big)[q] = 0. \tag{2.2}$$

This is the equation which we are going to consider. We shall consider certain objects related to it in a bounded domain. Physical phenomena which gave rise to the Dirac equation occur usually in unbounded domains but some of them (the Casimir effect, for instance) take place in bounded domains also. For more details see, e.g. [10].

2.2 Definition of Cauchy-Dirac-type integral

This section contains ideas and denotations from [12], where an analogue of the Cauchy-type integral for the time-harmonic bispinor fields theory was introduced, compare also with [10, Chapter 2] and [15].

The integral

$$K_{\mathbb{D}_{\omega,m}}[g](x) := -\int_{\Gamma} \check{\mathcal{K}}^{x}_{\mathbb{D}_{\omega,m}}[\sigma_{\mathbb{D}_{\omega,m}} g(\tau)], \quad x \notin \Gamma,$$

plays the role of the Cauchy-type integral in the theory of time-harmonic bispinor fields with $g : \Gamma \to \mathbb{C}^4$(see [12]) and we shall call it the Cauchy-Dirac-type integral, where

$$\sigma_{\mathbb{D}_{\omega,m}} := \frac{1}{2}\begin{pmatrix} (n_2 - in_1) & in_3 & in_3 & (n_2 + in_1) \\ -n_3 & i(n_2 + in_1) & i(n_2 - in_1) & -n_3 \\ -in_3 & -(n_2 + in_1) & (n_2 - in_1) & in_3 \\ -i(n_2 - in_1) & n_3 & -n_3 & i(n_2 + in_1) \end{pmatrix} dS,$$

$\vec{n}(\tau) = (n_1(\tau), n_2(\tau), n_3(\tau))$ is an outward pointing normal unit vector on Γ at $\tau \in \Gamma$, and dS is an element of the surface area in $\mathbb{R}^3$. The explicit form of time-harmonic relativistic Cauchy-Dirac kernel $\check{\mathcal{K}}^{x}_{\mathbb{D}_{\omega,m}}$ can be seen, e.g. in reference [12].

2.3 Some notations

Let $H_\mu(\Gamma, \mathbb{C}^3)$ denote the class of functions satisfying the Hölder condition $\{\vec{f} \in \mathbb{C}^3 |\ |\vec{f}(t_1) - \vec{f}(t_2)| \le L_f \cdot |t_1 - t_2|^\mu;\ \forall\{t_1, t_2\} \subset \Gamma,\ L_f = const\}$ with the exponent $0 < \mu \le 1$. Here $|\vec{f}|$ means the Euclidean norm in $\mathbb{C}^3 = \mathbb{R}^6$ while $|t|$ is the Euclidean norm in $\mathbb{R}^3$.

Let Γ be a surface in $\mathbb{R}^3$ which contains a finite number of conical points and a finite number of non-intersecting edges such that none of the edges contain any of conical points. If the complement (in Γ) of the union of conical points and edges, is a Liapunov surface, then we shall refer to Γ as a piece-wise Liapunov surface in $\mathbb{R}^3$. Let $\Omega = \Omega_+$ be a domain in $\mathbb{R}^3$ with the boundary Γ which is assumed to be a piece-wise Liapunov surface; denote $\Omega_- := \mathbb{R}^3 \setminus (\Omega_+ \cup \Gamma)$.

We will use a radiation condition for time-harmonic solutions of the Dirac equation:

$$\left(E_4 - i\gamma_1\gamma_2\gamma_3 \frac{x_\gamma}{|x|}\right) q(x) = o\left(\frac{1}{|x|}\right) \text{ as } |x| \to \infty \qquad (2.3)$$

when $\omega^2 \neq m^2$, or

$$q(x) = o\left(\frac{1}{|x|}\right) \text{ as } |x| \to \infty \tag{2.4}$$

when $\omega^2 = m^2$, uniformly for all directions $x/|x|$, where E_4 is the 4×4 identity matrix and $x_\gamma = \sum_{k=1}^{3} x_k \gamma_k$.

3 Basic facts of hyperholomorphic functional theory

In this section, we provide some background on quaternionic analysis needed in this paper. For more information, we refer the reader to [5] and to [10].

3.1 Basic facts of quaternions

We consider the skew-field of complex quaternions $\mathbb{H}(\mathbb{C})$:

$$\mathbb{H}(\mathbb{C}) := \{x = x_0 + i_1x_1 + i_2x_2 + i_3x_3;\ (x_0, x_1, x_2, x_3)^t \in \mathbb{C}^4\},$$

where the *imaginary units* i_1, i_2, i_3 satisfy the conditions

$$i_\ell i_k = -i_k i_\ell,\ \ell \neq k,$$

and

$$i_1i_2 = i_3,\ i_2i_3 = i_1,\ i_3i_1 = i_2.$$

The complex conjugation of $x = x_0 + i_1x_1 + i_2x_2 + i_3x_3$ is given by

$$Z_{\mathbb{C}}(x) := x^* = Re\ x - i\ Im\ x,$$

the quaternionic conjugation of $x = x_0 + i_1x_1 + i_2x_2 + i_3x_3$ is given by

$$Z_{\mathbb{H}}(x) := \bar{x} = x_0 - i_1x_1 - i_2x_2 - i_3x_3$$

and its norm by

$$|x| := \sqrt{x\bar{x}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}.$$

An important property is that

$$|xy| \leq \sqrt{2}|x| \cdot |y| \text{ for all } x, y \in \mathbb{H}(\mathbb{C}),$$

see [6].

3.2 Definition of Cauchy-type integral

Let $\lambda \in \mathbb{H}(\mathbb{C}) \setminus \{0\}$, and let α be its complex-quaternionic square root: $\alpha \in \mathbb{H}(\mathbb{C})$, $\alpha^2 = \lambda$. The function $f : \Omega \subset \mathbb{R}^3 \to \mathbb{H}(\mathbb{C})$ is called *left-α-hyperholomorphic* if

$$D_\alpha f := f\alpha + i_1 \frac{\partial}{\partial x_1} f + i_2 \frac{\partial}{\partial x_2} f + i_3 \frac{\partial}{\partial x_3} f = 0.$$

Because of the non-commutativity of the multiplication of quaternions there is also the dual notation of right-α-hyperholomorphic function. We set this article in the framework of the left-α-hyperholomorphic functions but all our results have counterparts for right-α-hyperholomorphic functions.

Let $\alpha \in \mathbb{H}(\mathbb{C})$ and let θ_α be the fundamental solution of the Helmholtz operator $\Delta_\lambda := \Delta + I\lambda$, where $\Delta := \sum_{k=1}^{3} \frac{\partial^2}{\partial x_k^2}$ and I is the identity operator. Then the fundamental solution of the operator D_α, $\mathcal{K}_\alpha$, is given by the formula (see [10]):

$$\mathcal{K}_\alpha(x) := -D_\alpha \theta_\alpha(x),$$

and its explicit form can be seen, e.g., in [16]. We shall use the notation $C^p(\Omega, \mathbb{H}(\mathbb{C}))$, $p \in \mathbb{N} \cup \{0\}$, which has the usual component-wise meaning. Denote by $\mathfrak{G}$ the set of zero divisors from $\mathbb{H}(\mathbb{C})$, i.e., $\mathfrak{G} := \{a \in \mathbb{H}(\mathbb{C}) \mid a \neq 0;\ \exists b \neq 0 :\ ab = 0\}$. Let $\sigma_\tau = \sum_{k=1}^{3} (-1)^{k-1} i_k dx_{[k]}$, where $dx_{[k]}$ denotes as usual the differential form $dx_1 \wedge dx_2 \wedge dx_3$ with the factor dx_k omitted. Let $\Omega = \Omega_+$ be a domain in $\mathbb{R}^3$ with the boundary Γ which is assumed to be a piece-wise Liapunov surface; denote $\Omega_- := \mathbb{R}^3 \setminus (\Omega_+ \cup \Gamma)$. If f is a Hölder function then its α-hyperholomorphic left Cauchy-type integral is defined (see [10, Subsection 4.16]):

$$K_\alpha[f](x) := -\int_\Gamma \check{\mathcal{K}}_\alpha^x[\sigma_\tau f(\tau)],\ x \in \Omega_\pm,$$

where

1. If $\alpha = \alpha_0 \in \mathbb{C}$, then
$$\check{\mathcal{K}}_\alpha^x[f](\tau) := \mathcal{K}_{\alpha_0}(x-\tau) f(\tau).$$

2. If $\alpha \notin \mathfrak{G}$, $\vec{\alpha}^2 \neq 0$, then
$$\check{\mathcal{K}}_\alpha^x[f](\tau) := \frac{1}{2\sqrt{\vec{\alpha}^2}} \mathcal{K}_{\xi_+}(x) f(\tau)(\sqrt{\vec{\alpha}^2} + \vec{\alpha}) + \frac{1}{2\sqrt{\vec{\alpha}^2}} \mathcal{K}_{\xi_-}(x) f(\tau)(\sqrt{\vec{\alpha}^2} - \vec{\alpha}). \tag{3.1}$$

3. If $\alpha \notin \mathfrak{G}$, $\vec{\alpha}^2 = 0$, then
$$\check{\mathcal{K}}_\alpha^x[f](\tau) := \mathcal{K}_{\alpha_0}(x) f(\tau) + \frac{\partial}{\partial \alpha_0}[\mathcal{K}_{\alpha_0}](x) f(\tau)\vec{\alpha}. \tag{3.2}$$

4. If $\alpha \in \mathfrak{G}$, $\alpha_0 \neq 0$, then

$$\check{\mathcal{K}}_\alpha^x[f](\tau) := \frac{1}{2\alpha_0}\mathcal{K}_{2\alpha_0}(x)f(\tau)\alpha + \frac{1}{2\alpha_0}\mathcal{K}_0(x)f(\tau)\overline{\alpha}. \tag{3.3}$$

5. If $\alpha \in \mathfrak{G}$, $\alpha_0 = 0$, then

$$\check{\mathcal{K}}_\alpha^x[f](\tau) := \mathcal{K}_0(x)f(\tau) + \theta_0(x)f(\tau)\alpha. \tag{3.4}$$

For more information about α-hyperholomorphic functions, we refer the reader to [5], [10], [13].
Setting

$$\begin{aligned} \breve{S}_\alpha[f](t) &:= \frac{2\pi - \gamma(t)}{2\pi} f(t) + S_\alpha[f](t), \\ P_\alpha[f](t) &:= \frac{1}{2}(f(t) + \breve{S}_\alpha[f](t)), \end{aligned}$$

where the singular integral $S_\alpha[f](t) := 2K_\alpha[f](t)$ exists in the sense of the Cauchy principal value; $\gamma(t)$ is the measure of a solid angle of the tangential conical surface at the point t or is the solid measure of the tangential dihedral angle at the point t.

4 Analogues of the Dirichlet problem for hyperholomorphic functions

As in the previous, we let Ω be the domain in $\mathbb{R}^3$ with a piece-wise Liapunov boundary and let $\alpha \in \mathbb{H}(\mathbb{C})$.

Interior Dirichlet problem 4.1 *Given an $\mathbb{H}(\mathbb{C})$-valued function g on $\Gamma := \partial\Omega$, the problem is to find a function $f \in H_\mu(\bar{\Omega}, \mathbb{H}(\mathbb{C})) \cap ker D_\alpha$ such that*

$$f|_\Gamma = g.$$

Solution. Let Γ as above and let $g \in H_\mu(\Omega, \mathbb{H}(\mathbb{C}))$. By [16, Theorem 2.3] Problem 4.1 is solvable if and only if

$$f = \breve{S}_\alpha. \tag{4.1}$$

Moreover, if (4.1) is fulfilled, then Problem 4.1 has the unique solution

$$f = K_\alpha[g].$$

By analogy, we immediately obtain the following

Exterior Dirichlet problem 4.2 *Find a function $f \in H_\mu(\bar{\Omega}_-, \mathbb{H}(\mathbb{C})) \cap ker D_\alpha$ such that*

$$f|_\Gamma = g.$$

For $|x| \to \infty$ it is required that f satisfies the radiation condition

$$\left(1 - i\frac{x}{|x|}\right) f(x) = o\left(\frac{1}{|x|}\right) \text{ as } |x| \to \infty;$$

when $\alpha = \alpha_0 \in \mathbb{C} \setminus \{0\}$, or when $\alpha \in \mathfrak{G}, \alpha_0 \neq 0$, or when $\alpha \notin \mathfrak{G}, \vec{\alpha}^2 = 0$, or when $\alpha \notin \mathfrak{G}, \vec{\alpha}^2 \neq 0$; or f satisfies the radiation condition

$$\left(1 \mp i\frac{x}{|x|}\right) f(x) = o\left(\frac{1}{|x|}\right) \text{ as } |x| \to \infty;$$

when$\alpha \in \mathbb{R} \setminus \{0\}$; or f satisfies the radiation condition

$$f(x) = o\left(\frac{1}{|x|}\right) \text{ as } |x| \to \infty;$$

when $\alpha \in \mathfrak{G}, \alpha_0 = 0$.

5 Analogues of the Dirichlet problem for time-harmonic spinor fields and MIT bag model on a pice-wise Liapunov surface

5.1 Analogues of the Dirichlet problem for Dirac equation

The following important proposition has been proved in paper [15].

Theorem 5.1 *(Extension of a given on Γ Hölder function up to a time-harmonic bispinor field)[15]. Let Ω be a bounded domain in $\mathbb{R}^3$ with the piece-wise Liapunov boundary.*

1. *In order for a function $g \in H_\mu(\Gamma, \mathbb{C}^4)$ to be a boundary value of a $\tilde{q}$ which is a solution of (2.2) in Ω_+ and is continuous in $\bar{\Omega}_+$, it is necessary and sufficient that*

$$g(t) = \breve{S}_{\mathbb{D}_{\omega,m}}[g](t), \ \forall t \in \Gamma \tag{5.1}$$

2. *In order for a function $g \in H_\mu(\Gamma, \mathbb{C}^4)$ to be a boundary value of $\tilde{q}$ which is a solution of (2.2) in $\bar{\Omega}_-$, and satisfies radiation condition (2.3) or (2.4), it is necessary and sufficient that*

$$g(t) = -\breve{S}_{\mathbb{D}_{\omega,m}}[g](t), \ \forall t \in \Gamma$$

Here $\breve{S}_{\mathbb{D}_{\omega,m}}[g](t) := \frac{2\pi-\gamma(t)}{2\pi}g(t)+S_{\mathbb{D}_{\omega,m}}[g](t)$ *and the singular integral* $S_{\mathbb{D}_{\omega,m}}[g](t) := 2K_{\mathbb{D}_{\omega,m}}[g](t)$ *exists in the sense of the Cauchy principal value.*

The Theorem 5.1 gives a complete description of solvability picture of the Dirichlet problem for equation (2.1):

Interior Dirichlet problem 5.1 *Find a function* $q \in H_\mu(\bar{\Omega}, \mathbb{C}^4)$ *which satisfies the Dirac equation (2.2) in* Ω *and the boundary condition*

$$q|_\Gamma = g$$

on Γ.

By Theorem 5.1 this problem is solvable if and only if condition (5.1) is satisfied. Moreover, if (5.1) is fulfilled, then the problem has the unique solution $q := K_{\mathbb{D}_{\omega,m}}$.

Similarly, from the Theorem 5.1, we have the following

Exterior Dirichlet problem 5.2 *Find a function* $q \in H_\mu(\bar{\Omega}_-, \mathbb{C}^4)$ *which satisfies the Dirac equation (2.2) in* Ω_- *and the boundary conditions*

$$q|_\Gamma = g$$

on Γ*. For* $|x| \to \infty$ *it is required that* q *satisfies the radiation condition (2.3) or (2.4).*

5.2 Function theory for the quaternionic Dirac operator

We start this Subsection with a brief description of the relations between the time-harmonic spinor fields theory and the theory of α-hyperholomorphic functions. One can find more about this in [10], [12]. The standard Dirac matrices have the well-known properties:

$$\gamma_0^2 = E_4, \quad \gamma_k^2 = -E_4, \quad k \in \mathbb{N}_3 := \{1,2,3\},$$
$$\gamma_j\gamma_k + \gamma_k\gamma_j = 0, \quad j,k \in \mathbb{N}_3^0 := \mathbb{N}_3 \cup \{0\}, \quad j \neq k,$$

where E_4 is the 4×4 identity matrix. The products of the Dirac matrices

$$\hat{i}_0 := E_4, \quad \hat{i}_1 := \gamma_3\gamma_2, \quad \hat{i}_2 := \gamma_1\gamma_3, \quad \hat{i}_3 := \gamma_1\gamma_2, \quad \hat{i} := \gamma_0\gamma_1\gamma_2\gamma_3,$$

have the following properties:

$$\hat{i}_0^2 = \hat{i}_0 = -\hat{i}_k^2, \quad \hat{i}_0\hat{i}_k = \hat{i}_k\hat{i}_0 = \hat{i}_k, \quad k \in \mathbb{N}_3,$$
$$\hat{i}_1\hat{i}_2 = -\hat{i}_2\hat{i}_1 = \hat{i}_3, \quad \hat{i}_2\hat{i}_3 = -\hat{i}_3\hat{i}_2 = \hat{i}_1, \quad \hat{i}_3\hat{i}_1 = -\hat{i}_1\hat{i}_3 = \hat{i}_2,$$
$$\hat{i} \cdot \hat{i}_k = \hat{i}_k \cdot \hat{i}, \quad k \in \mathbb{N}_3^0.$$

For $b \in \mathbb{H}(\mathbb{C})$, set

$$B_l(b) := \begin{pmatrix} b_0 & -b_1 & -b_2 & -b_3 \\ b_1 & b_0 & -b_3 & b_2 \\ b_2 & b_3 & b_0 & -b_1 \\ b_3 & -b_2 & b_1 & b_0 \end{pmatrix}.$$

Matrix subalgebra $\mathcal{B}_l(\mathbb{C}) := \{B_l(b) : b \in \mathbb{H}(\mathbb{C})\}$ and $\mathbb{H}(\mathbb{C})$ are isomorphic as complex algebras. Abusing a little we shall not distinguish, sometimes, between $B_l(b)$, the column $\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and the quaternion b. Set

$$\mathcal{D} := i\omega\gamma_0 - E_4\partial_1 - \gamma_1\partial_2 - \gamma_3\partial_3 + im.$$

We shall consider $\mathcal{D}$ on the set $C^1(\Omega, \mathcal{B}_l(\mathbb{C}))$ of corresponding matrices. Hence for us

$$\mathcal{D} : C^1(\Omega, \mathcal{B}_l(\mathbb{C})) \to C^0(\Omega, \mathcal{B}_l(\mathbb{C})).$$

In [10, Section 12] (see also [9, page 7563]) there was introduced the map $\mathcal{UA}$ which transforms a function $q : \tilde{\Omega} \subset \mathbb{R}^3 \to \mathbb{C}^4$ into the function $\rho : \Omega \subset \mathbb{R}^3 \to \mathbb{H}(\mathbb{C})$ by the rule:

$$\rho = \mathcal{UA}[q] := \frac{1}{2}[-(\tilde{q}_1 - \tilde{q}_2)i_0 + i(\tilde{q}_0 - \tilde{q}_3)i_1 - (\tilde{q}_0 + \tilde{q}_3)i_2 + i(\tilde{q}_1 + \tilde{q}_2)i_3],$$

where $\tilde{q}(x) := q(x_1, x_2, -x_3)$, the domain $\tilde{\Omega}$ is obtained from $\Omega \subset \mathbb{R}^3$ by the reflection $x_3 \to -x_3$. The corresponding inverse transform is given as follows:

$$(\mathcal{UA})^{-1}[\rho] = \mathcal{A}^{-1}\mathcal{U}^{-1}[\rho] := (-i\tilde{\rho}_1 - \tilde{\rho}_2, -\tilde{\rho}_0 - i\tilde{\rho}_3, \tilde{\rho}_0 - i\rho_3, i\tilde{\rho}_1 - \tilde{\rho}_2).$$

The maps $\mathcal{UA}$ and $(\mathcal{UA})^{-1}$ may be represented in a matrix form (see [10, Subsection 12.13]):

$$\rho = \mathcal{UA}[q] := \frac{1}{2}\begin{pmatrix} 0 & -1 & 1 & 0 \\ i & 0 & 0 & -i \\ -1 & 0 & 0 & -1 \\ 0 & i & i & 0 \end{pmatrix} \begin{pmatrix} \tilde{q}_0 \\ \tilde{q}_1 \\ \tilde{q}_2 \\ \tilde{q}_3 \end{pmatrix},$$

$$q = (\mathcal{UA})^{-1}[\rho] := \begin{pmatrix} 0 & -i & -1 & 0 \\ -1 & 0 & 0 & -i \\ 1 & 0 & 0 & -i \\ 0 & i & -1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\rho}_0 \\ \tilde{\rho}_1 \\ \tilde{\rho}_2 \\ \tilde{\rho}_3 \end{pmatrix}.$$

Direct computation leads to the equality

$$\mathbb{D}_{\omega,m} = -\gamma_0\hat{i}(\mathcal{UA})^{-1}\mathcal{D}\hat{i}_2(\mathcal{UA}), \tag{5.2}$$

on $C^1(\Omega, \mathbb{C}^4)$. Also we get $\mathcal{D}\hat{i}_2 = D_\alpha$, on $\mathcal{B}_l(\mathbb{C})$, where $\alpha := -(i\omega i_1 + m i_2)$. By these reasons $\mathcal{D}$ is termed "the quaternionic relativistic Dirac operator". Thus,

$$\ker \mathcal{D} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \ker D_\alpha.$$

There exists a one-to-one correspondence between elements of $\ker \mathcal{D}$ (which are matrices) and matrices of the form $B_l(q)$, with $q = q_0 i_0 + q_1 i_1 + q_2 i_2 + q_3 i_3$ being α-hyperholomorphic function.

5.3 Boundary integral criteria for the MIT bag model on a piece-wise Liapunov boundary

One of the most interesting problems in contemporary elementary particle physics is the problem of quark problem. At present there exist many modifications of the original, so-called MIT (Massachusetts Institute of Technology) bag model. The MIT bag model is an important tool for calculations, see for instance [1], [2] and [8].

Problem 5.3 *(MIT bag model). Let $\tilde{\Omega}$ be a bounded domain in $\mathbb{R}^3$ with piece-wise Liapunov boundary $\tilde{\Gamma}$ and $q \in C^1(\tilde{\Omega}) \cap C(\bar{\tilde{\Omega}})$. To find a solution to the Dirac equation in $\tilde{\Omega}$,*

$$\mathbb{D}_{\omega,m}[q] := (i\omega\gamma_0 - \sum_{k=1}^{3} \gamma_k \partial_k + im)[q] = 0, \tag{5.3}$$

satisfying the boundary condition

$$\sum_{k=1}^{3} \gamma_k \tilde{n}_k(x) q(x) = iq(x), \ x \in \tilde{\Gamma}, \tag{5.4}$$

where $\tilde{n}_k$ are the components of the unit outward normal to the boundary $\tilde{\Gamma}$ at the point x.

The following argument is pretty standard (e.g., see [10]).
A function q satisfies (5.3) if and only if its image $\rho := \mathcal{U}A[q]$ under the map $\mathcal{U}A$ satisfies the quaternionic equation

$$D_\alpha[\rho] = 0 \text{ in } \Omega, \tag{5.5}$$

where $\alpha := -(i\omega i_1 + m i_2)$. Applying the mapping $\mathcal{A}$ to both sides of (5.4) and combined with a straightforward calculation we rewrite the condition (5.4) as equality [10]

$$\frac{1}{2}(I + \vec{n} i M^{i_3} \mathbb{Z}_{\mathbb{C}})\mathcal{A}[q] = 0 \text{ on } \Gamma, \tag{5.6}$$

where $M^x f := f \cdot x, \ x \in \mathbb{H}(\mathbb{C})$. Setting

$$\begin{aligned} Q^{\pm} &:= \frac{1}{2}(I \mp \vec{n} i M^{i_3} Z_{\mathbb{C}}); \\ S^{\pm} &:= \frac{1}{2}(I \mp \vec{n} M^{i_2}), \end{aligned}$$

then the (5.6) can be rewritten as

$$Q^- \mathcal{A}[q] = 0 \text{ on } \Gamma. \tag{5.7}$$

The equality (5.7) may be represented into the boundary condition (see [10, Section 14]):

$$S^-[\rho] = 0 \text{ on } \Gamma.$$

Thus we have the biquaternionic reformulation of the Problem 5.3:

Problem 5.4 (*Biquaternionic MIT bag model problem*). *To find α-hyperholomorphic function $\alpha = -(i\omega i_1 + m i_2)$, with the boundary condition*

$$S^-[\rho] = 0 \text{ on } \Gamma.$$

[16, Theorem 2.3] gives a simple criterion of solvability of Problem 5.4.

Theorem 5.2 (*Solvability of the MIT bag model problem*). *Given $\rho \in H_\mu(\Gamma, \mathbb{H}(\mathbb{C}))$, Problem 5.4 is solvable if and only if*

$$\rho = P_\alpha[\rho] = S^+[\rho]$$

holds on Γ.

Theorem 5.3 *Given $\rho \in H_\mu(\Gamma, \mathbb{H}(\mathbb{C}))$, the function $\rho := K_\alpha[g]$ provides a solution to Problem 5.4 if and olny if*

$$S^- P_\alpha[g] = 0$$

holds on Γ.

Proof: The proof is fairly straightforward. Let $\rho := K_\alpha[g]$ be a solution to Problem 5.4. Note that by quaternionic Sokhotski-Plemelj formula [16, Theorem 2.1] we have

$$S^-[\rho]|_\Gamma = S^- P_\alpha[g] = 0.$$

Suppose now, on the contrary,

$$S^- P_\alpha[g] = 0 \text{ on } \Gamma.$$

Consider $\rho := K_\alpha[g]$. This satisfies (5.5) in Ω, and on Γ we may then write

$$\rho = P_\alpha[g].$$

Therefore $S^-[\rho] = 0$ on Γ. □

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