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On Positive Solutions of the Difference Equation $x_{n+1} = \frac{x_{n-5}}{1+x_{n-2}x_{n-5}}$

Ramazan Karatas

Selcuk University, Education Faculty
Mathematics Department, 42090, Meram Yeni Yol
Konya, Turkey
rkaratas@selcuk.edu.tr

Cengiz Cinar

Mathematics Department, Faculty of Education
Selcuk University, 42090, Konya, Turkey
ccinar@selcuk.edu.tr

Dağıstan Simsek

Selcuk University, Engineering-Architecture Faculty
Department of Industrial Engineering
42075 Kampüs/Konya, Turkey
dsimsek@selcuk.edu.tr

Abstract. We study the positive solutions and attractivity of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2}x_{n-5}} \quad \text{for} \quad n = 0, 1, 2, ...$$

where initial values are nonnegative real numbers.

Keywords: Difference Equation, Recursive Sequence, Positive Solutions, Equilibrium Point

1. INTRODUCTION

Recently there has been a great interest in studying the attractivity, the solutions and the periodic nature of non-linear difference equations. For example [1, 3-7, 9-11, 13-15]. In [8] Camouzis has investigated the global attractivity and local stability of the difference equation $x_{n+1} = \frac{x_{n-1}}{1+x_n x_{n-1}}$ for $n = 0, 1, 2, ...$ Also, in [2] Cinar has studied the positive solutions of this difference equation with the positive initial values. Morover, in [12] Stevic some additional information behavior of the solutions of this difference equation with the initial values $x_{-1}$and x_0 are real numbers.

Our aim in this paper is to investigate the positive solutions of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2}x_{n-5}} \quad \text{for} \quad n = 0, 1, 2, ... \tag{1.1}$$

where

$$x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1} \text{ and } x_0 \text{ are nonnegative real numbers.} \tag{1.2}$$

Similar to the references in this paper, we define Eq.(1.1) with (1.2) and investigate the solutions of this difference equation.

2. MAIN RESULTS

Theorem 1. *Assume that (1.2) holds and let* $\{x_n\}$ *be a solution of Eq.(1.1) with* $x_0 = a$, $x_{-1} = b$, $x_{-2} = c$, $x_{-3} = d, x_{-4} = e$, *and* $x_{-5} = f$. *Then for* $n = 0, 1, ...$ *all solutions of Eq.(1.1) are*

$$x_{6n+1} = \frac{f \prod_{i=1}^{n} (1+(2i)cf)}{(1+cf) \prod_{i=1}^{n} (1+(2i+1)cf)} ...(2.1), \quad x_{6n+2} = \frac{e \prod_{i=1}^{n} (1+(2i)be)}{(1+be) \prod_{i=1}^{n} (1+(2i+1)be)} ...(2.2),$$

$$x_{6n+3}=\frac{d\prod_{i=1}^{n}(1+(2i)da)}{(1+da)\prod_{i=1}^{n}(1+(2i+1)da)}\text{...(2.3)},\quad x_{6n+4}=\frac{c(1+cf)\prod_{i=1}^{n}((2i+1)cf+1)}{(1+2cf)\prod_{i=1}^{n}((2i+2)cf+1)}\text{...(2.4)}.$$

$$x_{6n+5}=\frac{b(1+be)\prod_{i=1}^{n}((2i+1)be+1)}{(1+2be)\prod_{i=1}^{n}((2i+2)be+1)}\text{...(2.5)},\quad x_{6n+6}=\frac{a(1+ad)\prod_{i=1}^{n}((2i+1)ad+1)}{(1+2ad)\prod_{i=1}^{n}((2i+2)ad+1)}\text{...(2.6)}$$

Proof. x_1,x_2,x_3,x_4,x_5 and x_6 are clear from Eq.(1.1). From our assumption (1.2) all solutions of Eq.(1.1) are positive. Also, for $n=1$ the resul holds. Now suppose that $n>1$ and our assumption holds for $(n-1)$. We shall show that the result holds for n. From our assumption for $(n-1)$, we have

$$x_{6n-5}=\frac{f\prod_{i=1}^{n-1}(1+(2i)cf)}{(1+cf)\prod_{i=1}^{n-1}(1+(2i+1)cf)},\quad x_{6n-4}=\frac{e\prod_{i=1}^{n-1}(1+(2i)be)}{(1+be)\prod_{i=1}^{n-1}(1+(2i+1)be)},$$

$$x_{6n-3}=\frac{d\prod_{i=1}^{n-1}(1+(2i)da)}{(1+da)\prod_{i=1}^{n-1}(1+(2i+1)da)},\quad x_{6n-2}=\frac{c(1+cf)\prod_{i=1}^{n-1}((2i+1)cf+1)}{(1+2cf)\prod_{i=1}^{n-1}((2i+2)cf+1)},$$

$$x_{6n-1}=\frac{b(1+be)\prod_{i=1}^{n-1}((2i+1)be+1)}{(1+2be)\prod_{i=1}^{n-1}((2i+2)be+1)}\text{ and }x_{6n}=\frac{a(1+ad)\prod_{i=1}^{n-1}((2i+1)ad+1)}{(1+2ad)\prod_{i=1}^{n-1}((2i+2)ad+1)}$$

Then, from (1.1) and above the equality, we have

$$x_{6n+1}=\frac{x_{6n-5}}{1+x_{6n-2}x_{6n-5}}=\frac{\frac{f\prod_{i=1}^{n-1}(1+(2i)cf)}{(1+cf)\prod_{i=1}^{n-1}(1+(2i+1)cf)}}{1+\frac{c(1+cf)\prod_{i=1}^{n-1}((2i+1)cf+1)}{(1+2cf)\prod_{i=1}^{n-1}((2i+2)cf+1)}.\frac{f\prod_{i=1}^{n-1}(1+(2i)cf)}{(1+cf)\prod_{i=1}^{n-1}(1+(2i+1)cf)}}$$

$$=\frac{\frac{f\prod_{i=1}^{n-1}(1+(2i)cf)}{(1+cf)\prod_{i=1}^{n-1}(1+(2i+1)cf)}}{1+\frac{cf\prod_{i=1}^{n-1}(1+(2i)cf)}{(1+2cf)\prod_{i=1}^{n-1}((2i+2)cf+1)}}=\frac{\frac{f\prod_{i=1}^{n-1}(1+(2i)cf)}{(1+cf)\prod_{i=1}^{n-1}(1+(2i+1)cf)}}{1+\frac{cf(1+2cf)\prod_{i=2}^{n-1}(1+(2i)cf)}{(1+2cf)\prod_{i=1}^{n-1}((2i+2)cf+1)}}=\frac{\frac{f\prod_{i=1}^{n-1}(1+(2i)cf)}{(1+cf)\prod_{i=1}^{n-1}(1+(2i+1)cf)}}{1+\frac{cf\prod_{i=2}^{n-1}(1+(2i)cf)}{\prod_{i=1}^{n-1}((2i+2)cf+1)}}$$

$$=\frac{\frac{f\prod_{i=1}^{n-1}(1+(2i)cf)}{(1+cf)\prod_{i=1}^{n-1}(1+(2i+1)cf)}}{1+\frac{cf\prod_{i=2}^{n-1}(1+(2i)cf)}{(2ncf+1)\prod_{i=2}^{n-1}((2i)cf+1)}}=\frac{f\prod_{i=1}^{n-1}(1+(2i)cf)}{(1+cf)\prod_{i=1}^{n-1}(1+(2i+1)cf)}.\frac{(2ncf+1)}{2ncf+cf+1}$$

$$=\frac{f\prod_{i=1}^{n}(1+(2i)cf)}{(1+cf)\prod_{i=1}^{n}(1+(2i+1)cf)}$$

Hence, we have

$$x_{6n+1}=\frac{f\prod\limits_{i=1}^{n}(1+(2i)cf)}{(1+cf)\prod\limits_{i=1}^{n}(1+(2i+1)cf)}$$

Smilarly,

$$x_{6n+4}=\frac{x_{6n-2}}{1+x_{6n+1}x_{6n-2}}=\frac{\frac{c(1+cf)\prod\limits_{i=1}^{n-1}((2i+1)cf+1)}{(1+2cf)\prod\limits_{i=1}^{n-1}((2i+2)cf+1)}}{1+\frac{c(1+cf)\prod\limits_{i=1}^{n-1}((2i+1)cf+1)}{(1+2cf)\prod\limits_{i=1}^{n-1}((2i+2)cf+1)}\cdot\frac{f\prod\limits_{i=1}^{n}(1+(2i)cf)}{(1+cf)\prod\limits_{i=1}^{n}(1+(2i+1)cf)}}$$

$$=\frac{\frac{c(1+cf)\prod\limits_{i=1}^{n-1}((2i+1)cf+1)}{(1+2cf)\prod\limits_{i=1}^{n-1}((2i+2)cf+1)}}{1+\frac{cf(1+2cf)\prod\limits_{i=1}^{n-1}((2i+1)cf+1)\prod\limits_{i=2}^{n}(1+(2i)cf)}{(1+2cf)(1+2ncf+cf)\prod\limits_{i=2}^{n}((2i)cf+1)\prod\limits_{i=1}^{n-1}(1+(2i+1)cf)}}$$

$$=\frac{\frac{c(1+cf)\prod\limits_{i=1}^{n-1}((2i+1)cf+1)}{(2cf+1)\prod\limits_{i=1}^{n-1}((2i+2)cf+1)}}{1+\frac{cf}{1+2ncf+cf}}=\frac{c(1+cf)\prod\limits_{i=1}^{n-1}((2i+1)cf+1)}{(2cf+1)\prod\limits_{i=1}^{n-1}((2i+2)cf+1)}\cdot\frac{1+2ncf+cf}{1+2ncf+2cf}$$

$$=\frac{\frac{c(1+cf)}{1+2ncf+cf}\prod\limits_{i=1}^{n}((2i+1)cf+1)}{\frac{(2cf+1)}{1+2ncf+2cf}\prod\limits_{i=1}^{n}((2i+2)cf+1)}\cdot\frac{1+2ncf+cf}{1+2ncf+2cf}$$

$$=\frac{c(1+cf)\prod\limits_{i=1}^{n}((2i+1)cf+1)}{(2cf+1)\prod\limits_{i=1}^{n}((2i+2)cf+1)}$$

Hence, we have

$$x_{6n+4}=\frac{c(1+cf)\prod\limits_{i=1}^{n}((2i+1)cf+1)}{(1+2cf)\prod\limits_{i=1}^{n}((2i+2)cf+1)}$$

Similarly, one can easily obtain (2.2), (2.3),(2.5) and (2.6). Thus, the proof is completed by indiction. ∎

Theorem 2. *Eq.(1.1) has a unique equilibrium point which is the number zero.*

Proof. For the equilibrium point of Eq.(1.1), we can write

$$\overline{x} = \overline{x}/(1 + \overline{x}.\overline{x}).$$

Then we have

$$\overline{x}^3 = 0.$$

Thus the equilibrium point of Eq.(1.1) is $\overline{x} = 0$. ∎

Corollary 3. *If* $0 \leq x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0 \leq 1$ *then* $0 \leq x_n \leq 1$ *for an arbitrary* $n(1, 2, ...)$.

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