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On the homogeneous ideal of general unions of simple and double points with support on a fixed plane curve

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Abstract. Fix an integral plane curve C. Here we study the postulation and the homogeneous ideal of general unions of u double and v simple points whose support is contained in C (at least when $\deg(C) \geq (3u+v)/2$).

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1. Schemes with supports on an integral plane curve

For any integer $m > 0$ let mP denote the fat point with multiplicity m of $\mathbf{P}^2$ with P as its support, i.e. the zero-dimensional subscheme of $\mathbf{P}^2$ with $(\mathcal{I}_P)^m$ as its ideal sheaf. It is very easy to check that general unions of fat points supported by a general plane curve of sufficiently high degree have good postulation and good minimal free resolution. Here we investigate the case in which the support lies on a fixed integral plane curve andthe multiplicity is at most 2. For background and standard results on the minimal free resolution of zero-dimensional subschemes of a projective space, see [1], [2] and [3]. We work over an algebraically closed field. First, we will state the case Z reduced and then the case $m_i \leq 2$ for all i.

Theorem 1. *Fix positive integer t, s, an integral degree t curve $C \subset \mathbf{P}^2$, and a general $S \subset C$ such that $\sharp(S) = s$. Let k be the minimal positive integer such that $s \leq (k+2)(k+1)/2$. Set $m := \lfloor k/2 \rfloor$ and $\epsilon := k - 2m$. Assume $t \geq k$ and $s \leq 4m + \epsilon$. Then S has the expected postulation and the expected minimal free resolution, i.e. $h^0(\mathbf{P}^2, \mathcal{I}_S(d)) = 0$ for all $d < k$, $h^0(\mathbf{P}^2, \mathcal{I}_S(d)) = (d+2)(d+1)/2 - s$ for all $d \geq k$ and the homogeneous ideal of S is generated by $(k+2)(k+1)/2 - s$ forms of degree k and $\max\{0, k(k+2) - 2s\}$ forms of degree $k+1$.*

Theorem 2. *Fix integer t, u, v such that $t > 0$, $u > 0$, $v \geq 0$, an integral degree t curve $C \subset \mathbf{P}^2$, and a general $P_1, \dots, P_u, Q_1 \dots, Q_v \in C$. Set $Z := 2P_1 \cup \dots \cup 2P_u \cup Q_1 \cup \dots \cup Q_v$. Let k be the minimal positive integer such that $3u + v \leq (k+2)(k+1)/2$. Set $m := \lfloor k/2 \rfloor$ and $\epsilon := k - 2m$. Assume $t > k$, $u \leq 2k+3$ and $3u + v \leq 4m - 4 + \epsilon$. Then Z has the expected postulation and the expected minimal free resolution, i.e.*

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$h^0(\mathbf{P}^2, \mathcal{I}_Z(d)) = 0$ for all $d < k$, $h^0(\mathbf{P}^2, \mathcal{I}_Z(d)) = (d+2)(d+1)/2 - 3u - v$ for all $d \geq k$ and the homogeneous ideal of Z is generated by $(k+2)(k+1)/2 - 3u - v$ forms of degree k and $\max\{0, k(k+2) - 6u - 2v\}$ forms of degree $k+1$.

Remark 1. From the dual of the Euler's sequence of $T\mathbf{P}^2$ we get $h^0(\mathbf{P}^2, \Omega^1{}_{\mathbf{P}^2}(t+1)) = t(t+2)$ for all $t \geq 2$, $h^0(\mathbf{P}^2, \Omega^1{}_{\mathbf{P}^2}(t+1)) = 0$ for all $t \leq 0$, $h^0(\mathbf{P}^2, \Omega^1{}_{\mathbf{P}^2}(2)) = 3$, $h^1(\mathbf{P}^2, \Omega^1{}_{\mathbf{P}^2}(t+1)) = 0$ for all $t \neq -1$ and $h^1(\mathbf{P}^2, \Omega^1{}_{\mathbf{P}^2}) = 1$.

Let $B \subset \mathbf{P}^2$ be a zero-dimensional scheme and t a positive integer. We recall that $h^1(\mathbf{P}^2, \mathcal{I}_B \otimes \Omega^1{}_{\mathbf{P}^2}(t+1)) = h^1(\mathbf{P}^2, \mathcal{I}_B(t)) = 0$ if and only if the minimal free resolution of B is in degree at most t (see e.g. [2]).

Let $D \subset \mathbf{P}^2$ be a smooth conic. Then $\Omega^1{}_{\mathbf{P}^2}|D$ is isomorphic to the direct sum of 2 line bundles of degree -31 ([**?**], Lemma 1.3). For all integers $m \geq 2$ the vector bundle $\Omega^1{}_{\mathbf{P}^2}(m)|D$ is isomorphic to the direct sum of 2 line bundles of degree $2m-3$. Hence $h^0(D, \mathcal{I}_{B,D} \otimes (\Omega^1{}_{\mathbf{P}^2}(m)|D)) = h^1(D, \mathcal{I}_{B,D} \otimes (\Omega^1{}_{\mathbf{P}^2}(m)|D)) = 0$ for every zero-dimensional scheme $B \subset D$ such that length$(B) = 2m-2$.

Proof of Theorem 1. Since the postulation of the schemes Z we will construct are easier, we will only write down that the homogeneous ideal of Z has the expected number of generators. By semicontinuity it sufficient to find one set $S \subset C$ satisfying all the conditions claimed in the statement of Theorem 1. Let $A_1, \dots, A_m$ be m general smooth conics. Hence $\sharp(C \cap A_i) = 2t$ for all i and $C \cap A_i \cap A_j = \emptyset$ for all $i \neq j$. Take $S_i \subset C \cap D_i$ such that $\sharp(S \cap D_i) = 2k + 2 - 2i$. If k is even, then set $S := S_1 \cup \dots \cup S_m$. Use Remark 1 with respect to the conic D_i and the integer $k+2-2i$, $1 \leq i \leq m$, and apply m times Horace Lemma. If k is odd, then take any $Q \in C \backslash (S_1 \cup \dots \cup S_m)$ and set $S := \{Q\} \cup S_1 \cup \dots \cup S_m$. Do the same proof. □

Proof of Theorem 2. Since the postulation of the schemes Z we will construct are easier, we will only write down that the homogeneous ideal of Z has the expected number of generators. We will modify the proof of Theorem 1. We first do the case $u = 2k-3$ and $t > k$. Let D, R be smooth conics intersecting transversally C and such that $\sharp(D \cap R) = \sharp(D \cap R \cap C) = 4$. By assumption $\sharp(C \cap (D \cup R)) = 2t - 4$. We take as $P_1, \dots, P_4$ the points of $C \cap D \cap R)$, while we take exactly $k-4$ (resp. $k-3$) of the other points in $C \cap D$ (resp. $C \cap R$), say $P_5, \dots, P_k$ (resp. $P_{k+1}, \dots, P_{2k-3}$. Set $W := 2P_1 \cup \dots \cup 2P_{2k-3}$. Hence length$(D \cap W) = 2k$. Use Remark 1 with respect to D and Horace Lemma. Notice that $\mathrm{Res}_D(W) = \{P_1, \dots, P_4\} \cup 2P_{k+1} \cup \dots \cup 2P_{2k-3}$. Hence length$(\mathrm{Res}_D(W) \cap R) = 2k-2$. We apply Horace lemmma with respect to R. We have $\mathrm{Res}_R(\mathrm{Res}_D(W)) = \{P_1, \dots, P_{k-5}, P_{k+1}, \dots, P_{2k-3}\}$. There are $m-2$ smooth conics B_j, $1 \leq j \leq m-2$, such that their union contains $\{P_1, \dots, P_{k-5}, P_{k+1}, \dots, P_{2k-3}$. For general D, R we may also arrange to have $C \cap B_i \cap B_j = \emptyset$ for all $i \neq j$. Then we insert the v simple points among the remaining points of $C \cap (B_1 \cup \dots \cup B_{m-2}$. If $u < 2k-3$, then we just use $P_1, \dots, P_u$ and insert more simple points. □

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