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MULTI-COMPONENT RETURNS TO SCALE: A DEA-BASED APPROACH

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Abstract

Efficiency modeling in Data Envelopment Analysis (DEA) has generally been studied to derivation of a single measure of efficiency. In the current paper, efficiency is viewed from a multi-component perspective. We first, define a multi-component efficiency model. Then, we define "returns to scale" in multi-component environments and propose a method for identifying the nature of returns to scale. An illustrative application of the methodology to a sample of bank branches is given.

Keywords: Data Envelopment Analysis, returns to scale.

1 Introduction

Data Envelopment Analysis(DEA) introduced by Charnes et al.[3] presented a constant returns to scale(RTS) model for evaluating the performance of a set of co9mparable decision making units(DMU). DEA has been used in a very wide range of applications to measure the efficiency of organizational units such as schools or bank branches when there are multiple in-commensurate inputs and outputs. In these applications, it is necessary to identify the nature of returns to scale which characterizes efficient production. In most real applications, a single measure of efficiency and single estimation of returns to scale provided by DEA methodology has been an adequate and useful means of comparing units and identifying best performance. Some methods have been developed to

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test the nature of aggregate returns to scale. However, in most real situations, the Decision Making Unit (DMU) involved may perform several different functions or can be separated into different components. In such cases, inputs, in particular resources are often shared among those components. The purpose of this paper is to develop the returns to scale concept to the multi-component environments and to demonstrate the fact that the nature of returns to scale, i.e. increasing, constant and decreasing, may be different in components of a DMU. Therefore, it is an important research subject to investigate the nature of returns to scale for each components of a special DMU_p. The paper is structured as follows: The next section of the paper gives a summary of basic DEA models. The third section of the paper, presents a multi-component efficiency measurement. A method for estimating returns to scale for each components of a DMU is given in section four. Section five applies the proposed approach to a real data set involving 14 bank branches. Conclusions appear in section six.

2 Background

The data domain for a DEA study is the set A of n data points, $a^1, a^2, \ldots, a^n$; one for each DMU. Each data point composed of two types of components, those pertaining to the m inputs, $0 \neq x_j \geq 0$, and those corresponding to the s outputs, $0 \neq y_j \geq 0$. We organize the data in the following way:

$$A = \left[\ a^1, a^2, \ldots, a^n\ \right], \text{ where, } \quad a^j = \left[\begin{array}{c} Y_j \\ -X_j \end{array} \right]; \quad j = 1, 2, \ldots, n,$$

and A is the $m + s$ by n matrix the columns of which are the data points.

2.1 CCR model

The production possibility set T_c of the CCR model (Charnes et al.[3]) is defined as

$$T_c = \{(x, y) \mid x \geq \sum_{j=1}^{n} \lambda_j x_j, \quad y \leq \sum_{j=1}^{n} \lambda_j y_j, \quad \lambda_j \geq 0,\ j = 1, 2, \ldots, n\}$$

To estimate a CCR-efficiency score of the specific *p-th* DMU, we use the following original DEA model (CCR ratio form):

$$\begin{array}{l} Max \quad e_p = \sum_{r=1}^{s} u_r y_{rp} \\ \text{subject to :} \\ \sum_{i=1}^{m} v_i x_{ip} = 1, \\ \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \quad j = 1, \ \ldots, \ n, \\ u_r \geq \epsilon, \quad r = 1, \ \ldots, \ s, \\ v_i \geq \epsilon, \quad i = 1, \ \ldots, \ m, \end{array} \tag{1}$$

where $\epsilon > 0$ is a non-archimedean construct. This model is an input-oriented program. The dual version of (1) is

$$\begin{array}{l} Min\ \ \theta_p - \ \epsilon[\sum_{i=1}^{m} s_i{}^- + \ \sum_{r=1}^{s} s_r{}^+] \\ \text{subject to :} \\ \sum_{j=1}^{n} \lambda_j x_{ij} + \ s_i^- \ = \ \theta_p x_{ip}, \quad i = 1, \ldots, m, \\ \sum_{j=1}^{n} \lambda_j y_{rj} - \ s_r^+ \ = \ y_{rp}, \quad r = 1, \ldots, s, \\ \lambda_j, s_i{}^-, s_r{}^+ \geq 0, \ \ for\ all\ i, j, r. \end{array} \tag{2}$$

Obviously, the optimal value to both programs can not exceed one. If the optimal value is equal to one, then a particular DMU_p is located on the CCR frontier.

2.2 BCC model

The production possibility set T_v of the BCC model (Banker et al.[1]) is defined as

$$T_v = \{(x, y) \mid x \geq \sum_{j=1}^{n} \lambda_j x_j, \ \ y \leq \sum_{j=1}^{n} \lambda_j y_j, \ \ \sum_{j=1}^{n} \lambda_j = 1, \ \ \lambda_j \geq 0, \ j = 1, 2, \ldots, n\}$$

To estimate a BCC-efficiency score of the specific *p-th* DMU, we use the following model (BCC-ratio form):

$$\begin{array}{l} Max\ \ e_p = \sum_{r=1}^{s} u_r y_{rp} - \rho \\ \text{subject to :} \\ \sum_{i=1}^{m} v_i x_{ip} = 1, \\ \sum_{r=1}^{s} u_r y_{rj} - \rho - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \qquad j = 1, \ \ldots, \ n, \\ u_r \geq \epsilon, \qquad r = 1, \ \ldots, \ s, \\ v_i \geq \epsilon, \qquad i = 1, \ \ldots, \ m, \end{array} \tag{3}$$

where ρ is un-restricted in sign. This model represents variable returns to scale (VRS) by the addition of the constraint $\sum_{j=1}^{n} \lambda_j = 1$ in contrast to CCR model which exhibits constant returns to scale. The dual version of (3) is

$$\begin{array}{l} Min\ \ \theta_p - \ \epsilon[\sum_{i=1}^{m} s_i{}^- + \ \sum_{r=1}^{s} s_r{}^+] \\ \text{subject to :} \\ \sum_{j=1}^{n} \lambda_j x_{ij} + \ s_i^- \ = \ \theta_p x_{ip}, \quad i = 1, \ldots, m, \\ \sum_{j=1}^{n} \lambda_j y_{rj} - \ s_r^+ \ = \ y_{rp}, \quad r = 1, \ldots, s, \\ \sum_{j=1}^{n} \lambda_j = 1, \\ \lambda_j, s_i{}^-, s_r{}^+ \geq 0, \ \ for\ all\ i, j, r. \end{array} \tag{4}$$

If the optimal value is equal to one, then a particular DMU_p is located on the BCC frontier.

2.3 Returns to Scale

To estimate the nature of returns to scale, Banker and Thrall [2] proposed the an approach as follows:
(i) constant returns to scale prevail for DMU_p *iff* $\rho = 0$ *in some optimal solution to (1),*
(ii) increasing returns to scale prevail for DMU_p *iff* $\rho < 0$ *in all optimal solutions to (1),*
(iii) decreasing returns to scale prevail for DMU_p *iff* $\rho > 0$ *in all optimal solutions to (1),* ,(see[2]).
Toward this end, solve the following two programs for each BCC-efficient DMU_p, :

$$\begin{array}{cc} Max\ \rho^{+} = \rho & Min\ \rho^{-} = \rho \\ \text{subject } to: & \text{subject } to: \\ \sum_{r=1}^{s} u_r y_{rp} + \rho = 1, & \sum_{r=1}^{s} u_r y_{rp} + \rho = 1, \\ \sum_{i=1}^{m} v_i x_{ip} = 1, & \sum_{i=1}^{m} v_i x_{ip} = 1, \\ \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \rho \leq 0, \forall j, & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \rho \leq 0, \forall j, \\ u_r \geq \epsilon, \quad r = 1, \ldots, s, & u_r \geq \epsilon, \quad r = 1, \ldots, s, \\ v_i \geq \epsilon, \quad i = 1, \ldots, m, & v_i \geq \epsilon, \quad i = 1, \ldots, m, \end{array} \tag{5}$$

(i) constant returns to scale prevail for DMU_p *iff* $\rho^{+*} \geq 0$ *or* $\rho^{-*} le0$*,*
(ii) increasing returns to scale prevail for DMU_p *iff* $\rho^{+*} < 0$*,*
(iii) decreasing returns to scale prevail for DMU_p *iff* $\rho^{-*} > 0$.

3 Multi-component performance

For notational purposes, let $Y_k^{(1)}, Y_k^{(2)}, \ldots, Y_k^{(b)}$ denote the set of each components transactions of DMU_k. Also, let $X_k^{(1)}, X_k^{(2)}, \ldots, X_k^{(b)}$ denote $I_1, I_2, \ldots, I_b$-dimensional vectors of dedicated inputs to each components and ${X_k}^{(c)}$ is a I_c-dimensional vector of shared inputs. All components are involved in producing the J_c-dimensional vector of outputs ${Y_k}^{(c)}$. Some portion α_i of the shared inputs $X_k^{(c)}$ is allocated to the *i-th* component. Also, *i-th* component is involved in producing some portion β_i of the shared outputs ${Y_k}^{(c)}$. (Note that $\alpha_i \geq 0, \beta_i \geq 0$ and $\sum_{i=1}^{b} \alpha_i = \sum_{i=1}^{b} \beta_i = 1$). In proposed model α_i and β_i are decision variables which must be determined. Now, by the above mentioned discussion, it is evident that the outputs $Y_k^{(1)}, Y_k^{(2)}, \ldots, Y_k^{(b)}, Y_k^{(c)}$ are produced from the inputs $X_k^{(1)}, X_k^{(2)}, \ldots, X_k^{(b)}, X_k^{(c)}$. As the measures proposed by Jahanshahloo et al. [5], a measure of aggregate performance $e_k^{(a)}$ can be represented by:

$$e_k^{(a)} = \frac{\sum_{i=1}^{b} [{\mu^{(i)}}^T Y_k^{(i)} + {\mu^{(s_i)}}^T (\beta_i {Y_k}^{(c)}) + \rho_i]}{[\sum_{i=1}^{b} [{v^{(i)}}^T X_k^{(i)} + {v^{(s_i)}}^T (\alpha_i {X_k}^{(c)})]}$$

For this representation, the vectors μ and v would be determined in a DEA manner to be discussed below. From $e_k^{(a)}$, performance measure for each component of DMU_k can be represented by:

$$e_k^{(i)} = \frac{{\mu^{(i)}}^T Y_k^{(i)} + {\mu^{(s_i)}}^T(\beta_i Y_k^{(c)}) + \rho_i}{{v^{(i)}}^T X_k^{(i)} + {v^{(s_i)}}^T(\alpha_i X_k^{(c)})}; \quad i = 1, \ldots, b.$$

It is easy to show that the aggregate performance measure $e_k^{(a)}$ is a convex combination of $e_k^{(i)}$s. To derive $e_k^{(a)}$ and $e_k^{(i)}$, we simply maximize the aggregate performance measure $e_k^{(a)}$ subject to the constraints $e_k^{(a)}$ and $e_k^{(i)}$ are less than or equal to one. Consider the following mathematical programming:

$$\begin{array}{l} Max \;\; e_k^{(a)} \\ \text{subject to :} \\ e_j^{(i)} \leq 1, \quad i = 1, \ldots, b, \; j = 1, \ldots, n, \\ \sum_{i=1}^b \alpha_i = \sum_{i=1}^b \beta_i = 1, \\ \mu^{(i)}, \mu^{(s_i)}, v^{(i)}, v^{(s_i)} \geq \epsilon, \quad i = 1, \ldots, b, \\ \alpha_i, \beta_i \geq 0, \quad i = 1, \ldots, b. \end{array} \tag{6}$$

Model (6) is ratio and proceeding in the manner of Charnes and Cooper (1962), this model can be expressed in following non-ratio form:

$$\begin{array}{l} Max \;\; \sum_{i=1}^b [{\mu^{(i)}}^T Y_k^{(i)} + {\mu^{(s_i)}}^T(\beta_i Y_k^{(c)}) + \rho_i] \\ \text{subject to :} \\ \sum_{i=1}^b [{v^{(i)}}^T X_k^{(i)} + {v^{(s_i)}}^T(\alpha_i X_k^{(c)})] = 1, \\ {\mu^{(i)}}^T Y_j^{(i)} + {\mu^{(s_i)}}^T(\beta_i Y_j^{(s_i)}) + \rho_i - {v^{(i)}}^T X_j^{(i)} - {v^{(s_i)}}^T(\alpha_i X_j^{(c)} \leq 0, \quad \forall i, \; j \\ \sum_{i=1}^b \alpha_i = \sum_{i=1}^b \beta_i = 1, \\ \mu^{(i)}, \mu^{(s_i)}, v^{(i)}, v^{(s_i)} \geq \epsilon, \quad i = 1, \ldots, b, \\ \alpha_i, \beta_i \geq 0, \quad i = 1, \ldots, b. \end{array} \tag{7}$$

Since α_i and β_i are decision variables, this problem is clearly nonlinear. If we make the change of variables $\bar{\mu}^{(s_i)} = \mu^{(s_i)}\beta_i;\; i = 1, \ldots, b$ and $\bar{v}^{(s_i)} = v^{(s_i)}\alpha_i;\; i = 1, \ldots, b$, (7) reduces to the following form:

$$\begin{array}{l} Max \;\; \sum_{i=1}^b [{\mu^{(i)}}^T Y_k^{(i)} + {\bar{\mu}^{(s_i)}}^T Y_k^{(c)} + \rho_i] \\ \text{subject to :} \\ \sum_{i=1}^b [{v^{(i)}}^T X_k^{(i)} + {\bar{v}^{(s_i)}}^T X_k^{(c)}] = 1, \\ {\mu^{(i)}}^T Y_j^{(i)} + {\bar{\mu}^{(s_i)}}^T Y_j^{(c)} + \rho_i - {v^{(i)}}^T X_j^{(i)} - {\bar{v}^{(s_i)}}^T X_j^{(c)} \leq 0, \quad \forall i, \; \forall j, \\ \mu^{(i)}, V^{(i)} \geq \epsilon, \quad i = 1, \ldots, b, \\ \bar{\mu}^{(s_i)} \geq \beta_i \epsilon, \quad i = 1, \ldots, b, \\ \bar{v}^{(s_i)} \geq \alpha_i \epsilon, \quad i = 1, \ldots, b. \end{array} \tag{8}$$

Again we point out that this model is almost similar to that developed by Cook et al.[4] and Jahanshahloo et al. [5]. In that case, they considered constant returns to scale.

4 Multi-component Returns to Scale

From the above mentioned discussion, we can estimate the nature of returns to scale for each components of DMU_k. First, we derive $e_k^{(a)}$, the aggregate efficiency of DMU_k, by (8). The modified returns to scale estimation technique, solves the following two programs for each efficient component t:

$$\begin{array}{l} Max \;\; {\rho_t}^{+} = \rho_t, \\ \text{subject to :} \\ \sum_{i=1}^{b} [{\mu^{(i)}}^T Y_k^{(i)} + {\bar{\mu}^{(s_i)}}^T Y_k^{(c)} + \rho_i = e_k^{(a)} \\ \\ {\mu^{(t)}}^T Y_k^{(t)} + {\bar{\mu}^{(s_t)}}^T Y_k^{(c)} + \rho_t = 1, \\ \sum_{i=1}^{b} [{v^{(i)}}^T X_k^{(i)} + {\bar{v}^{(s_i)}}^T X_k^{(c)}] = 1, \\ {\mu^{(i)}}^T Y_j^{(i)} + {\bar{\mu}^{(s_i)}}^T Y_j^{(c)} + \rho_i - {v^{(i)}}^T X_j^{(i)} - {\bar{v}^{(s_i)}}^T X_j^{(c)} \leq 0, \quad \forall i,\; j \\ \mu^{(i)}, v^{(i)} \geq \epsilon, \quad i = 1, \ldots, b, \\ \bar{\mu}^{(s_i)} \geq \beta_i \epsilon, \quad i = 1, \ldots, b, \\ \bar{v}^{(s_i)} \geq \alpha_i \epsilon, \quad i = 1, \ldots, b. \end{array} \tag{9}$$

$$\begin{array}{l} Min \;\; {\rho_t}^{-} = \rho_t, \\ \text{subject to :} \\ \sum_{i=1}^{b} [{\mu^{(i)}}^T Y_k^{(i)} + {\bar{\mu}^{(s_i)}}^T Y_k^{(c)} + \rho_i = e_k^{(a)} \\ \\ {\mu^{(t)}}^T Y_k^{(t)} + {\bar{\mu}^{(s_t)}}^T Y_k^{(c)} + \rho_t = 1, \\ \sum_{i=1}^{b} [{v^{(i)}}^T X_k^{(i)} + {\bar{v}^{(s_i)}}^T X_k^{(c)}] = 1, \\ {\mu^{(i)}}^T Y_j^{(i)} + {\bar{\mu}^{(s_i)}}^T Y_j^{(c)} + \rho_i - {v^{(i)}}^T X_j^{(i)} - {\bar{v}^{(s_i)}}^T X_j^{(c)} \leq 0, \quad \forall i,\; j \\ \mu^{(i)}, v^{(i)} \geq \epsilon, \quad i = 1, \ldots, b, \\ \bar{\mu}^{(s_i)} \geq \beta_i \epsilon, \quad i = 1, \ldots, b, \\ \bar{v}^{(s_i)} \geq \alpha_i \epsilon, \quad i = 1, \ldots, b. \end{array} \tag{10}$$

There are three cases as follows:
(i) constant returns to scale prevail for component t iff ${\rho_t}^{+} \geq 0$ *and* ${\rho_t}^{-} \leq 0$*,*
(ii) increasing returns to scale prevail for component t iff ${\rho_t}^{+} < 0$*,*
(iii) decreasing returns to scale prevail for component t iff ${\rho_t}^{-} > 0$*.*

5 An Example

The model proposed in this paper, is applied to a real data set on 14 Iranian commercial bank branches. Each branch is separated into two different components as: Sales and Services. The data set consists of two inputs and three outputs. All inputs are shared among these two components and all components are involved in producing output 3. The chosen input and output

Table 1. Input and output measures used in the application

Component	Inputs	Outputs
Sales	Input1, Input2	Output1, Output3
Services	Input1, Input2	Output2, Output3

Table 2. The data summary for Iranian bank branches

DMU_j	Input 1	Input 2	Output 1	Output 2	Output 3
1	104	12.37730346	9814.2	3836	598.2
2	232	10.66387289	12189.4	2099	189.4
3	75	6.882153269	1418.6	819	5418.6
4	365	12.64855327	75324.8	6681	1534.8
5	74	6.101590769	6504	352	4654
6	364	10.82581519	20048.6	3178	8048.6
7	271	15.45777769	7124.8	3631	3124.8
8	274	11.81160135	8228.4	1405	828.4
9	98	13.04011423	9151.6	9365	3951.6
10	22	9.101575962	47847.8	275	7847.8
11	75	5.0457675	886.6	748	1886.6
12	646	24.54479462	27732.4	3248	7732.4
13	54	11.25436077	651.4	5365	11251.4
14	281	12.2868675	227.8	1573	2227.8

measures that are used in this application are summarized in table 1. The relevant data are displayed in table 2. Running the method discussed in section 3 yield to the results that are summarized in table 3. The second and third columns of the table display the nature of RTS for two components. As the table indicates, constant returns to scale prevail for DMU_{10} and DMU_{11} in both components.

6 Conclusions

Estimating returns to scale in Data envelopment analysis provides important information on scale efficiency and on improving the performance of DMUs. In the current paper, we developed the DEA technique to one that determines multi-component returns to scale. An illustrative application of the methodology to a sample of bank branches is given.

Table 3. The RTS decomposition

DMU_j	Component 1	Component 2
DMU_1	DRS	DRS
DMU_2	DRS	DRS
DMU_3	DRS	DRS
DMU_4	CRS	DRS
DMU_5	DRS	DRS
DMU_6	DRS	DRS
DMU_7	DRS	DRS
DMU_8	DRS	DRS
DMU_9	DRS	CRS
DMU_{10}	CRS	CRS
DMU_{11}	DRS	DRS
DMU_{12}	DRS	DRS
DMU_{13}	CRS	CRS
DMU_{14}	DRS	DRS

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