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The Method of Generalized Quasilinearization and Higher Order of Convergence for Second-Order Boundary Value Problems

Ramzi S. N. Alsaedi

Department of Mathematics
King Abdul Aziz University
Jeddah P.O. Box 80203, Saudi Arabia
ramzialsaedi@yahoo.co.uk

Abstract

The generalized quasilinearization method for second-order boundary value problem has been extended when the forcing function is the sum of two functions without require that any of the two functions involved to be 2-hyperconvex or 2-hyperconcave. Two sequences are developed under suitable conditions which converge to the unique solution of the boundary value problem. Furthermore, the convergence obtain here is of order 3.

1. Introduction

The method of quasilinearization [1] combined with the technique of lower and upper solutions is an excellent tool for solving a large class of nonlinear problems. This technique works fruitfully only for the problems involving convex/concave functions. Later after that the convexity assumption was relaxed and the method was generalized and extended in various directions to make it applicable to a large class of problems. It has referred to as a generalized quasilinearization method, see [8]. The method is extremely useful in scientific computations due to its accelerated rat of convergence as in [9, 10].

In [3, 14], the authors have obtained a higher order of convergence for initial value problems. They considered situations when the forcing function is either hyperconvex or hyperconcave. In [11], the authors have obtained the results of higher order of convergence for first order initial value problems when the forcing function is the sum of hyperconvex and hyperconcave functions with natural and coupled lower and upper solutions. In [12], the authors have obtained the results of higher order of convergence for second-oredr boundary value problems when the forcing function is the sum of 2-hyperconvex and 2-hyperconcave functions with natural and coupled lower and upper solutions. This paper extend and genralized the result of [12] for the second-order boundary value problems to make it applicable to a large class of problems, by taking the forcing function to be the sum of two functions without require any of the two functions involved to be 2-hyperconvex or 2-hyperconcave. The author has proved the existence of the unique solution of the nonlinear problem using natural lower and upper solutions. The author demonstrate the iterates converge cubically to the unique solution of the nonlinear problem. Merely the result related to coupled lower and upper solutions stated without proof due to monotony.

2. Preliminaries

Consider the following second-order boundary value problem

$$-u'' = f(t,u) + g(t,u), \quad Bu(\mu) = b_\mu, \quad \mu = 0,1, \ t \in J = [0,1], \tag{2.1}$$

where $Bu(\mu) = \tau_\mu u(\mu) + (-1)^{\mu+1}\nu_\mu u'(\mu) = b_\mu$, $\tau_\mu, \tau_\mu \geq 0, \tau_\mu + \tau_\mu > 0$, $\nu_\mu, \nu_\mu > 0$, $b_\mu \in R$ and $f, g \in C[\Omega, R]$ and $\Omega = J \times R$.

Here some definitions and notations will be given to facilitate later explanations.

Definition 2.1. The functions $\alpha_0, \beta_0 \in C^2[J,R]$ are said to be natural lower and upper solutions of (2.1) if

$$-\alpha_0'' \leq f(t,\alpha_0) + g(t,\alpha_0), \quad B\alpha_0(\mu) \leq b_\mu \quad \text{on } J,$$

$$-\beta_0'' \geq f(t,\beta_0) + g(t,\beta_0), \quad B\beta_0(\mu) \geq b_\mu \quad \text{on } J. \tag{2.2}$$

Definition 2.2. A function $h : A \to B$, $A, B \subset R$ is called m-hyperconvex, $m \geq 0$, if $h \in C^{m+1}[A,B]$ and $d^{m+1}h/du^{m+1} \geq 0$ for $u \in A$; h is called m-hyperconcave if the inequality is reversed.

In this paper, we use the maximum norm of u over J, that is,

$$\| u \| = \max\{| u(t) | : t \in J\}.$$

Also throughout this paper the following notation

$$\frac{\partial^k f(t,u)}{\partial u^k} = f^{(k)}(t,u)$$

has been used for any function $f(t,u)$ and for $k = 0,1,2$.

Now, let us recall some well known theorem and corollaries which we need in our main results relative to the BVP

$$-u'' = f(t,u,u'), \quad Bu(\mu) = b_\mu, \quad \mu = 0,1, \ t \in J = [0,1], \tag{2.3}$$

where $Bu(\mu) = \tau_\mu u(\mu) + (-1)^{\mu+1}\nu_\mu u'(\mu) = b_\mu$, $\tau_\mu, \tau_\mu \geq 0, \tau_\mu + \tau_\mu > 0$, $\nu_\mu, \nu_\mu > 0$, $b_\mu \in R$ and $f \in C[J \times R \times R, R]$. For details see [2, 5, 6].

Theorem 2.3. Assume that

(A_1) $\alpha_0, \beta_0 \in C^2[J, R]$ are lower and upper solutions of (2.3).

(A_2) $f_u, f_{u'}$ exist and are continuous, $f_u < 0$ and $f_u \neq 0$ on $\Omega = [(t, u, u') : t \in J,\ \alpha_0 \leq u \leq \beta_0]$ and $u' = \alpha_0'(t) = \beta_0'(t)$.

Then we have $\alpha_0(t) \leq \beta_0(t)$ on J.

As a special case of the above theorem which is known as the maximum principl, when u'-term is missing, is given by

Corollery 2.4. Let $q, r \in C[J, R]$ with $r(t) \geq 0$ on J. Suppose further that $p \in C^2[J, R]$ and

$$-p'' \leq -rp, \quad Bp(\mu) \leq 0 \quad \text{on } J. \tag{2.4}$$

Then $p(t) \leq 0$ on J. If the inequalities are reversed, then $p(t) \geq 0$ on J.

The next corollary is a special case of [8, Theorem 3.1.3].

Corollery 2.5. Assume that $\alpha_0, \beta_0 \in C^2[J, R]$ are lower and upper solutions of (2.1) respectively such that $\alpha_0(t) \leq \beta_0(t)$ on J. Then there exists a solution u for the BVP (2.1) such that $\alpha_0 \leq u \leq \beta_0$ on J.

3. Main results

Consider the following BVP

$$-u'' = f(t, u) + g(t, u), \quad Bu(\mu) = b_\mu, \quad \mu = 0, 1,\ t \in [0, 1], \tag{3.1}$$

where $Bu(\mu) = \tau_\mu u(\mu) + (-1)^{\mu+1}\nu_\mu u'(\mu) = b_\mu$, $\tau_\mu, \tau_\mu \geq 0, \tau_\mu + \tau_\mu > 0$, $\nu_\mu, \nu_\mu > 0$, $b_\mu \in R$ and $f, g \in C[\Omega, R]$ and $\Omega = [(t, u) \in J \times R : \alpha_0(t) \leq u(t) \leq \beta_0(t)]$, and $\alpha_0, \beta_0 \in C^2[J, R]$ with $\alpha_0(t) \leq \beta_0(t)$ on J,

Theorem 3.1. Assume that

(A_1) $\alpha_0, \beta_0 \in C^2[J, R]$ are lower and upper solutions of (3.1), respectively, such that $\alpha_0(t) \leq \beta_0(t)$ on J,

(A_2) $f, g \in C^3[\Omega, R]$ such that $f(t, u)$ is nondecreasing, $g(t, u)$ is nonincreasing and $f_u(t, u) + g_u(t, u) < 0$ for every $(t, u) \in \Omega$.

Then there exists monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$, $n \geq 0$ which converges uniformly and monotonically to the unique solution of (3.1) and the convergence is of order 3.

Proof. Set

$$\phi(t, u) = F(t, u) - f(t, u) \ \ ; \ \ \psi(t, u) = G(t, u) - g(t, u) \quad \text{on} \ \ \Omega, \tag{3.2}$$

where $F, G \in C^3[\Omega, R]$ such that F is a 2-hyperconvex function in u and G is a 2-hyperconcave function in u on J [i.e., $F^{(3)}(t, u) \geq 0,\ G^{(3)}(t, u) \leq 0$ for $(t, u) \in \Omega$].

In view of $F^{(3)}(t,u) \geq 0$, for $(t,u) \in \Omega$, we see that

$$F(t,x) \geq \sum_{i=0}^{2} \frac{F^{(i)}(t,y)(x-y)^i}{i!}, \quad x \geq y, \tag{3.3}$$

$$F(t,x) \leq \sum_{i=0}^{2} \frac{F^{(i)}(t,y)(x-y)^i}{i!}, \quad x \leq y. \tag{3.4}$$

Similarly, in view of $G^{(3)}(t,u) \leq 0$ for $(t,u) \in \Omega$, we have

$$G(t,x) \geq \sum_{i=0}^{1} \frac{G^{(i)}(t,y)(x-y)^i}{i!} + \frac{G^{(2)}(t,x)(x-y)^2}{2!}, \quad x \geq y, \tag{3.5}$$

$$G(t,x) \leq \sum_{i=0}^{1} \frac{G^{(i)}(t,y)(x-y)^i}{i!} + \frac{G^{(2)}(t,x)(x-y)^2}{2!}, \quad x \leq y. \tag{3.6}$$

Therefore, (3.3), (3.4), (3.5) and (3.6) can be written in following form

$$f(t,x) \geq f(t,y) + \sum_{i=1}^{2} \frac{F^{(i)}(t,y)(x-y)^i}{i!} - [\phi(t,x) - \phi(t,y)],$$

$$x \geq y, \tag{3.7}$$

$$f(t,x) \leq f(t,y) + \sum_{i=1}^{2} \frac{F^{(i)}(t,y)(x-y)^i}{i!} - [\phi(t,x) - \phi(t,y)],$$

$$x \leq y, \tag{3.8}$$

$$g(t,x) \geq g(t,y) + G^{(1)}(t,y)(x-y) + \frac{G^{(2)}(t,x)(x-y)^2}{2!}$$

$$-[\psi(t,x) - \psi(t,y)], \quad x \geq y, \tag{3.9}$$

$$g(t,x) \leq g(t,y) + G^{(1)}(t,y)(x-y) + \frac{G^{(2)}(t,x)(x-y)^2}{2!}$$

$$-[\psi(t,x)-\psi(t,y)], \quad x \le y, \tag{3.10}$$

respectively. Let first consider the following BVPs:

$$\begin{aligned}-w'' &= \chi(t,\alpha,\beta;w)\\ &= f(t,\alpha)+\sum_{i=1}^{2}\frac{F^{(i)}(t,\alpha)(w-\alpha)^i}{i!}-[\phi(t,w)-\phi(t,\alpha)]\\ &+g(t,\alpha)+G^{(1)}(t,\alpha)(w-\alpha)+\frac{G^{(2)}(t,\beta)(w-\alpha)^2}{2!}\\ &-[\psi(t,w)-\psi(t,\alpha)],\end{aligned}$$

$$Bw(\mu)=b_\mu \quad \text{on } J\ ; \tag{3.11}$$

$$\begin{aligned}-v'' &= \omega(t,\alpha,\beta;v)\\ &= f(t,\beta)+\sum_{i=1}^{2}\frac{F^{(i)}(t,\beta)(v-\beta)^i}{i!}-[\phi(t,v)-\phi(t,\beta)]\\ &+g(t,\beta)+G^{(1)}(t,\beta)(v-\beta)+\frac{G^{(2)}(t,\alpha)(v-\beta)^2}{2!}\\ &-[\psi(t,v)-\psi(t,\beta)],\end{aligned}$$

$$Bv(\mu)=b_\mu \quad \text{on } J\ ; \tag{3.12}$$

Now by using the above BVPs (3.11) and (3.12) to develop the sequences$\{\alpha_n\}$ and $\{\beta_n\}$ respectively. Initially, to prove (α_n,β_0) are the lower and upper solutions of (3.11) and (3.12) respectively, let us consider natural lower and upper solutions of the equation (3.1):

$$-\alpha_0'' \le f(t,\alpha_0)+g(t,\alpha_0), \quad B\alpha_0(\mu)\le b_\mu \quad \text{on } J, \tag{3.13}$$

$$-\beta_0'' \ge f(t,\beta_0)+g(t,\beta_0), \quad B\beta_0(\mu)\ge b_\mu \quad \text{on } J,$$

where $\alpha_0(t)\le\beta_0(t)$. The inequalities (3.7), (3.9) and (3.13) imply

$$\begin{aligned}-\alpha_0'' &\le f(t,\alpha_0)+g(t,\alpha_0)\\ &= \chi(t,\alpha_0,\beta_0;\alpha_0), \quad B\alpha_0(\mu)\le b_\mu,\\ -\beta_0'' &\ge f(t,\beta_0)+g(t,\beta_0)\end{aligned}$$

$$\ge f(t,\alpha_0)+\sum_{i=1}^{2}\frac{F^{(i)}(t,\alpha_0)(\beta_0-\alpha_0)^i}{i!}-[\phi(t,\beta_0)-\phi(t,\alpha_0)] \tag{3.14}$$

$$+g(t,\alpha_0)+G^{(1)}(t,\alpha_0)(\beta_0-\alpha_0)+\frac{G^{(2)}(t,\beta_0)(\beta_0-\alpha_0)^2}{2!}$$
$$-[\psi(t,\beta_0)-\psi(t,\alpha_0)]$$
$$=\chi(t,\alpha_0,\beta_0;\beta_0),\qquad B\beta_0(\mu)\geq b_\mu.$$

By apply *Corollary 2.5* together (3.14) conclude that there exists a solution $\alpha_1(t)$ of (3.11) with $\alpha=\alpha_0$ and $\beta=\beta_0$ such that $\alpha_0(t)\leq\alpha_1(t)\leq\beta_0(t)$ on J.

Using the inequalities (3.8), (3.10) and (3.13), we can get

$$-\beta_0''\geq f(t,\beta_0)+g(t,\beta_0)$$
$$=\omega(t,\alpha_0,\beta_0;\beta_0),\quad B\beta_0(\mu)\geq b_\mu,$$
$$-\alpha_0''\leq f(t,\alpha_0)+g(t,\alpha_0)$$

$$\leq f(t,\beta_0)+\sum_{i=1}^{2}\frac{F^{(i)}(t,\beta_0)(\alpha_0-\beta_0)^i}{i!}-[\phi(t,\alpha_0)-\phi(t,\beta_0)] \tag{3.15}$$

$$+g(t,\beta_0)+G^{(1)}(t,\beta_0)(\alpha_0-\beta_0)+\frac{G^{(2)}(t,\alpha_0)(\alpha_0-\beta_0)^2}{2!}$$
$$-[\psi(t,\alpha_0)-\psi(t,\beta_0)]$$
$$=\omega(t,\alpha_0,\beta_0;\alpha_0)\qquad B\alpha_0(\mu)\leq b_\mu.$$

Hence α_0,β_0 are lower and upper solutions of (3.12) with $\alpha_0(t)\leq\beta_0(t)$. Apply *Corollary 2.5* together (3.15) conclude that there exists a solution $\beta_1(t)$ of (3.12) with $\alpha=\alpha_0$ and $\beta=\beta_0$ such that $\alpha_0(t)\leq\beta_1(t)\leq\beta_0(t)$ on J.

Now to prove that $\alpha_1(t)$ is the unique solution of (3.11), we need to prove that $\partial\chi(t,\alpha_0,\beta_0;\alpha_1)/\partial\alpha_1<0$. Since $F(t,u)$ is a 2-hyperconvex function in u and $G(t,u)$ is a 2-hyperconcave function in u on J with $f_u(t,u)+g_u(t,u)<0$ on Ω, we have

$$\frac{\partial\chi(t,\alpha_0,\beta_0;\alpha_1)}{\partial\alpha_1}=f^{(1)}(t,\alpha_1)-\frac{F^{(3)}(t,\xi_1)(\alpha_1-\alpha_0)^2}{2}$$
$$+g^{(1)}(t,\alpha_1)+G^{(3)}(t,\eta_1)(\alpha_1-\alpha_0)(\beta_0-\xi_2)$$

$$\leq f^{(1)}(t,\alpha_1)+g^{(1)}(t,\alpha_1)<0, \tag{3.16}$$

where $\alpha_0\leq\xi_1$, $\xi_2\leq\alpha_1$ and $\xi_2\leq\eta_1\leq\beta_0$. Hence by the special case of *Theorem 2.3* with u'-term missing, we can conclude that α_1 is the unique solution of (3.11). Similarly, one can prove that β_1 is the unique solution of (3.12).

Using the nonincreasing property of $G^{(2)}(t,u)$, (3.7), (3.8), (3.9) and (3.10) with $\alpha_0(t)\leq\alpha_1(t)\leq\beta_0(t)$, $\alpha_0(t)\leq\beta_1(t)\leq\beta_0(t)$ we have

$$-\alpha_1''=\chi(t,\alpha_0,\beta_0;\alpha_1)$$

$$= f(t,\alpha_0) + \sum_{i=1}^{2} \frac{F^{(i)}(t,\alpha_0)(\alpha_1-\alpha_0)^i}{i!} - [\phi(t,\alpha_1)-\phi(t,\alpha_0)]$$

$$+g(t,\alpha_0) + G^{(1)}(t,\alpha_0)(\alpha_1-\alpha_0) + \frac{G^{(2)}(t,\beta_0)(\alpha_1-\alpha_0)^2}{2!}$$

$$-[\psi(t,\alpha_1)-\psi(t,\alpha_0)]$$

$$\leq f(t,\alpha_1)+g(t,\alpha_1), \qquad B\alpha_1(\mu) \leq b_\mu; \tag{3.17}$$

$$-\beta_1'' = \omega(t,\alpha_0,\beta_0;\beta_1)$$

$$= f(t,\beta_0) + \sum_{i=1}^{2} \frac{F^{(i)}(t,\beta_0)(\beta_1-\beta_0)^i}{i!} - [\phi(t,\beta_1)-\phi(t,\beta_0)]$$

$$+g(t,\beta_0) + G^{(1)}(t,\beta_0)(\beta_1-\beta_0) + \frac{G^{(2)}(t,\alpha_0)(\beta_1-\beta_0)^2}{2!}$$

$$-[\psi(t,\beta_1)-\psi(t,\beta_0)]$$

$$\geq f(t,\beta_1)+g(t,\beta_1), \qquad B\beta_1(\mu) \geq b_\mu. \tag{3.18}$$

Since α_1, β_1 are lower and upper solutions of (3.1), we can apply the special case of *Theorem 2.3* to obtain $\alpha_1 \leq \beta_1$ on J. Thus we have $\alpha_0 \leq \alpha_1 \leq \beta_1 \leq \beta_0$ on J.

Assume now that α_n and β_n are solutions of BVPs (3.11) and (3.12), respecively, with $\alpha = \alpha_{n-1}$ and $\beta = \beta_{n-1}$ such that $\alpha_{n-1} \leq \alpha_n \leq \beta_n \leq \beta_{n-1}$ on J and

$$-\alpha_n'' \leq f(t,\alpha_n)+g(t,\alpha_n), \quad B\alpha_n(\mu) \leq b_\mu \quad \text{on } J,$$

$$-\beta_n'' \geq f(t,\beta_n)+g(t,\beta_n), \quad B\beta_n(\mu) \geq b_\mu \quad \text{on } J. \tag{3.19}$$

We need to show that $\alpha_n \leq \alpha_{n+1} \leq \beta_{n+1} \leq \beta_n$ on J, where α_{n+1} and β_{n+1} are solutions of BVPs (3.11) and (3.12), respecively, with $\alpha = \alpha_n$ and $\beta = \beta_n$.

The inequalities (3.7), (3.9) and (3.19) imply

$$-\alpha_n'' \leq f(t,\alpha_n)+g(t,\alpha_n)$$

$$= \chi(t,\alpha_n,\beta_n;\alpha_n), \quad B\alpha_n(\mu) \leq b_\mu,$$

$$-\beta_n'' \geq f(t,\beta_n)+g(t,\beta_n)$$

$$\geq f(t,\alpha_n) + \sum_{i=1}^{2} \frac{F^{(i)}(t,\alpha_n)(\beta_n-\alpha_n)^{(i)}}{i!} - [\phi(t,\beta_n)-\phi(t,\alpha_n)] \tag{3.20}$$

$$+g(t,\alpha_n)+G^{(1)}(t,\alpha_n)(\beta_n-\alpha_n)+\frac{G^{(2)}(t,\beta_n)(\beta_n-\alpha_n)^{(2)}}{2!}$$
$$-[\psi(t,\beta_n)-\psi(t,\alpha_n)]$$
$$=\chi(t,\alpha_n,\beta_n;\beta_n),\qquad B\beta_n(\mu)\geq b_\mu.$$

This prove that α_n,β_n are lower and upper solutions of (3.11) with $\alpha=\alpha_n$ and $\beta=\beta_n$. Hence using (3.20) and *Corollary 2.5* we can conclude that there exists a solution α_{n+1} of (3.11) with $\alpha=\alpha_n$ and $\beta=\beta_n$ such that $\alpha_n\leq\alpha_{n+1}\leq\beta_n$ on J.

The inequalities (3.8), (3.10) and (3.19) imply

$$-\beta_n''\geq f(t,\beta_n)+g(t,\beta_n)$$
$$=\omega(t,\alpha_n,\beta_n;\beta_n),\quad B\beta_n(\mu)\geq b_\mu,$$
$$-\alpha_n''\leq f(t,\alpha_n)+g(t,\alpha_n)$$
$$\leq f(t,\beta_n)+\sum_{i=1}^{2}\frac{F^{(i)}(t,\beta_n)(\alpha_n-\beta_n)^{(i)}}{i!}-[\phi(t,\alpha_n)-\phi(t,\beta_n)] \tag{3.21}$$
$$+g(t,\beta_n)+G^{(1)}(t,\beta_n)(\alpha_n-\beta_n)+\frac{G^{(2)}(t,\alpha_n)(\alpha_n-\beta_n)^{(2)}}{2!}$$
$$-[\psi(t,\alpha_n)-\psi(t,\beta_n)]$$
$$=\omega(t,\alpha_n,\beta_n;\alpha_n)\qquad B\alpha_n(\mu)\leq b_\mu.$$

Hence α_n,β_n are lower and upper solutions of (3.12) with $\alpha=\alpha_n$ and $\beta=\beta_n$. Applying *Corollary 2.5* we can conclude that there exists a solution β_{n+1} of (3.12) with $\alpha=\alpha_n$ and $\beta=\beta_n$ such that $\alpha_n\leq\beta_{n+1}\leq\beta_n$ on J. In view of assumptions on f and g, α_n,β_n are unique by the special case of *Theorem 2.3.*

Furthermore, by (3.7), (3.8), (3.9) and (3.10) with $\alpha_n\leq\alpha_{n+1}\leq\beta_n$, $\alpha_n\leq\beta_{n+1}\leq\beta_n$, and $G^{(2)}(t,u)$ nonincreasing in u, we have

$$\begin{aligned}-\alpha_{n+1}'' &= \chi(t,\alpha_n,\beta_n;\alpha_{n+1})\\ &= f(t,\alpha_n)+\sum_{i=1}^{2}\frac{F^{(i)}(t,\alpha_n)(\alpha_{n+1}-\alpha_n)^{(i)}}{i!}-[\phi(t,\alpha_{n+1})-\phi(t,\alpha_n)]\end{aligned}$$
$$+g(t,\alpha_n)+G^{(1)}(t,\alpha_n)(\alpha_{n+1}-\alpha_n)+\frac{G^{(2)}(t,\beta_n)(\alpha_{n+1}-\alpha_n)^{(2)}}{2!}$$
$$-[\psi(t,\alpha_1)-\psi(t,\alpha_0)]$$
$$\leq f(t,\alpha_{n+1})+g(t,\alpha_{n+1}),\qquad B\alpha_{n+1}(\mu)\leq b_\mu; \tag{3.22}$$

$$-\beta''_{n+1} = \omega(t, \alpha_n, \beta_n; \beta_{n+1})$$

$$= f(t, \beta_n) + \sum_{i=1}^{2} \frac{F^{(i)}(t, \beta_n)(\beta_{n+1} - \beta_n)^{(i)}}{i!} - [\phi(t, \beta_{n+1}) - \phi(t, \beta_n)]$$

$$+g(t, \beta_n) + G^{(1)}(t, \beta_n)(\beta_{n+1} - \beta_n) + \frac{G^{(2)}(t, \alpha_n)(\beta_{n+1} - \beta_n)^{(2)}}{2!}$$

$$-[\psi(t, \beta_{n+1}) - \psi(t, \beta_n)]$$

$$\geq f(t, \beta_{n+1}) + g(t, \beta_{n+1}), \qquad B\beta_{n+1}(\mu) \geq b_\mu. \tag{3.23}$$

Since $\alpha_{n+1}, \beta_{n+1}$ are lower and upper solutions of (3.1), we can apply the special case of *Theorem 2.3* to obtain $\alpha_{n+1} \leq \beta_{n+1}$ on J. This proves $\alpha_n \leq \alpha_{n+1} \leq \beta_{n+1} \leq \beta_n$ on J. Thus by induction, we have

$$\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq \beta_n \leq \cdots \leq \beta_1 \leq \beta_0 \quad \text{on } J.$$

By the fact that α_n, β_n are lower and upper solutions of (3.1) with $\alpha_n \leq \beta_n$ and *Corollary 2.5* we can conclude that there exists a solution $u(t)$ of (3.1) such that $\alpha_n \leq u \leq \beta_n$ on J. So we have

$$\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq u \leq \beta_n \leq \cdots \leq \beta_1 \leq \beta_0 \quad \text{on } J. \tag{3.24}$$

Using Green's function, we can write $\alpha_n(t)$ and $\beta_n(t)$ as follows:

$$\alpha_n(t) = \int_0^1 K(t, s)\chi(s, \alpha_{n-1}(s), \beta_{n-1}(s); \alpha_n(s))ds,$$

$$\beta_n(t) = \int_0^1 K(t, s)\omega(s, \alpha_{n-1}(s), \beta_{n-1}(s); \beta_n(s))ds. \tag{3.25}$$

Here $K(t, s)$ is the Green's function given by

$$\frac{1}{c}x(s)y(t), \qquad 0 \leq s \leq t \leq 1,$$

$$\frac{1}{c}x(t)y(s), \qquad 0 \leq t \leq s \leq 1, \tag{3.26}$$

where $x(t) = (\tau_0/\nu_0)t + 1$ and $y(t) = (\tau_1/\nu_1)(1 - t) + 1$ are two linear independent solutions of the homogenuos equation $-u'' = 0$ and $c = x(t)y'(t) - x'(t)y(t)$. We can prove that the sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ are equicontnuous and uniformly bounded. Now applying Ascoli-Arzela's theorem, we can show that there exist subsequences $\{\alpha_{n,j}(t)\}$ and $\{\beta_{n,j}(t)\}$, such that $\alpha_{n,j}(t) \to \rho(t)$ and $\beta_{n,j}(t) \to r(t)$ with $\rho(t) \le u(t) \le r(t)$ on J. Since the sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ are monotone, we have $\alpha_n(t) \to \rho(t)$ and $\beta_n(t) \to r(t)$. Taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \alpha_n(t) = \rho(t) \le u(t) \le r(t) = \lim_{n\to\infty} \beta_n(t).$$

Next we show that $r(t) \le \rho(t)$. From BVPs (3.11) and (3.12) we get

$$-\rho(t)'' = f(t, \rho(t)) + g(t, \rho(t)), \quad B\rho(\mu) = b_\mu,$$

$$-r(t)'' = f(t, r(t)) + g(t, r(t)), \quad Br(\mu) = b_\mu. \tag{3.27}$$

Set $p(t) = r(t) - \rho(t)$ and note that $Bp(\mu) = 0$. we have

$$-p''(t) = -r''(t) - (-\rho''(t))$$

$$= f(t, r(t)) + g(t, r(t)) - f(t, \rho(t)) - g(t, \rho(t))$$

$$= f_u(t, \xi)(r(t) - \rho(t)) + g_u(t, \eta)(r(t) - \rho(t))$$

$$= (f_u(t, \xi) + g_u(t, \eta))p, \tag{3.28}$$

where ξ, η are between r and ρ. This implies that $-p'' \le -kp$, where $f_u + g_u \le -k < 0$. Now applying *Corollary 2.4* we get $r(t) \le \rho(t)$ on J. This proves $r(t) = \rho(t) = u(t)$. Hence $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ converge uniformly and monotonically to the unique solution of (3.1).

Let us consider the order of convergence of $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ to the unique solution $u(t)$ of (3.1). To obtain this, set

$$p_n(t) = u(t) - \alpha_n(t) \ge 0,$$

$$q_n(t) = \beta_n(t) - u(t) \ge 0, \tag{3.29}$$

for $t \in J$ with $Bp_n(\mu) = Bq_n(\mu) = 0$. therefore, we can write

$$p_{n+1}(t) = \int_0^1 K(t, s)[f(s, u) + g(s, u) - \chi(s, \alpha_n(s), \beta_n(s); \alpha_{n+1}(s))]ds, \tag{3.30}$$

where $K(t, s)$ is the Green's function given by (3.26).

Now using the Taylor series expansion with Lagrange remainder, the mean value theorem together with (A_2) of the hypothesis and the properties on F and G, we obtain

$$\begin{aligned}
0 &\le p_{n+1}(t)\\
&= \int_0^1 K(t,s)\{f(s,u)+g(s,u)-[f(t,\alpha_n)+g(t,\alpha_n)\\
&\quad +\sum_{i=1}^{2}\frac{F^{(i)}(t,\alpha_n)(\alpha_{n+1}-\alpha_n)^{(i)}}{i!}-[\phi(t,\alpha_{n+1})-\phi(t,\alpha_n)]\\
&\quad +G^{(1)}(t,\alpha_n)(\alpha_{n+1}-\alpha_n)+\frac{G^{(2)}(t,\beta_n)(\alpha_{n+1}-\alpha_n)^{(2)}}{2!}\\
&\quad -[\psi(t,\alpha_{n+1})-\psi(t,\alpha_n)]]\}ds\\
&= \int_0^1 K(t,s)\{f(s,u)+g(s,u)-[f(t,\alpha_{n+1})+g(t,\alpha_{n+1})\\
&\quad -\frac{F^{(3)}(t,\xi_1)(\alpha_{n+1}-\alpha_n)^{(3)}}{3!}-\frac{G^{(2)}(t,\xi_2)(\alpha_{n+1}-\alpha_n)^{(2)}}{2!}\\
&\quad +\frac{G^{(2)}(t,\beta_n)(\alpha_{n+1}-\alpha_n)^{(2)}}{2!}]\}ds\\
&\le \int_0^1 K(t,s)\{f_u(s,\eta_1)(u-\alpha_{n+1})+g_u(s,\eta_2)(u-\alpha_{n+1})\\
&\quad +\frac{F^{(3)}(t,\xi_1)(u-\alpha_n)^{(3)}}{3!}-\frac{G^{(3)}(t,\eta_3)(\beta_n-\xi_2)(u-\alpha_n)^{(2)}}{2!}\}ds\\
&= \int_0^1 K(t,s)\{[f_u(s,\eta_1)+g_u(s,\eta_2)]p_{n+1}+\frac{F^{(3)}(t,\xi_1){p_n}^{(3)}}{3!}\\
&\quad -\frac{G^{(3)}(t,\eta_3)(q_n+p_n){p_n}^{(2)}}{2!}\}ds,
\end{aligned}$$

where $\alpha_n \le \xi_1,\ \xi_2 \le \alpha_{n+1} \le \eta_1,\ \eta_2 \le u$ and $\xi_2 \le \eta_3 \le \beta_n$. Let $\mid K(t,s)\mid \le A_1$, $\mid f_u(s,u)+g_u(s,\nu)\mid \le A_2$, $\mid F^{(3)}(t,u)/3!\mid \le A_3$ and $\mid G^{(3)}(t,u)/2!\mid \le A_4$. Then we have

$$\| p_{n+1}\| \le k_1 \| p_n\|^3 + k_2 \| p_n\|^2 (\| q_n\| + \| p_n\|), \tag{3.31}$$

where $k_1 = A_1A_3/(1-A_1A_2)$ and $k_2 = A_1A_4/(1-A_1A_2)$.

Similarly, we can write

$$q_{n+1}(t) = \int_0^1 K(t,s)[\omega(t,\alpha_n,\beta_n;\beta_{n+1}) - f(s,u) - g(s,u)]ds, \tag{3.32}$$

where $K(t,s)$ is the Green's function given by (3.26).

Using the Taylor series expansion with Lagrange remainder, the mean value theorem together with (A_2) of the hypothesis and the properties on F and G, we can show

$$\| q_{n+1} \| \leq k_1 \| q_n \|^3 + k_2 \| q_n \|^2 (\| q_n \| + \| p_n \|), \tag{3.33}$$

where $k_1 = A_1A_3/(1 - A_1A_2)$ and $k_2 = A_1A_4/(1 - A_1A_2)$.

Hence combining (3.31) and (3.33) we obtain

$$\max_{t\in J} \mid u(t) - \alpha_{n+1}(t) \mid + \max_{t\in J} \mid \beta_{n+1}(t) - u(t) \mid$$

$$\leq C[\max_{t\in J} \mid u(t) - \alpha_n(t) \mid + \max_{t\in J} \mid \beta_n(t) - u(t) \mid]^3, \tag{3.34}$$

where C is an appropriate positive constant. This completes the proof. □

We note that the unique solution we have obtained is the unique solution of (3.1) in the sector determined by the lower and upper solutions.

Next we merely state a result without proof using coupled lower and upper solutions of (3.1). However, in order to show the existence of the unique solution of the iterates, we use the existence result [6, Theorem 2.4.1] for systems and special case of the comparison theorem of [6].

Theorem 3.2. Assume that

(A_1) $\alpha_0, \beta_0 \in C^2[J, R]$ are coupled lower and upper solutions of (3.1), respectively, with $\alpha_0(t) \leq \beta_0(t)$ on J such that

$$-\alpha_0'' \leq f(t, \beta_0) + g(t, \alpha_0), \quad B\alpha_0(\mu) \leq b_\mu \quad \text{on } J,$$

$$-\beta_0'' \geq f(t, \alpha_0) + g(t, \beta_0), \quad B\beta_0(\mu) \geq b_\mu \quad \text{on } J, \tag{3.35}$$

(A_2) $f, g \in C^3[\Omega, R]$ such that $f(t,u)$, $g(t,u)$ are nonincreasing and $f_u(t,u) - g_u(t,u) > 0$ for every $(t,u) \in \Omega$ and

$$f_u(t,u) \leq -\max_{\Omega}[F^{(3)}(t,u)](\beta_0 - \alpha_0)^2 \leq 0 \quad \text{on } \Omega.$$

Then there exists monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$, $n \geq 0$ such that

$$\begin{aligned}
-\alpha_n'' &= f(t, \beta_{n-1}) + F^{(1)}(t, \beta_{n-1})(\beta_n - \beta_{n-1}) + \frac{F^{(2)}(t, \alpha_{n-1})(\beta_n - \beta_{n-1})^2}{2!} \\
&\quad -[\phi(t, \beta_n) - \phi(t, \beta_{n-1})] + g(t, \alpha_{n-1}) + G^{(1)}(t, \alpha_{n-1})(\alpha_n - \alpha_{n-1}) \\
&\quad + \frac{G^{(2)}(t, \beta_{n-1})(\alpha_n - \alpha_{n-1})^2}{2!} - [\psi(t, \alpha_n) - \psi(t, \alpha_{n-1})] \\
B\alpha_n(\mu) &= b_\mu;
\end{aligned}$$

$$
\begin{aligned}
-\beta_n'' &= f(t,\alpha_{n-1}) + F^{(1)}(t,\alpha_{n-1})(\alpha_n - \alpha_{n-1}) + \frac{F^{(2)}(t,\beta_{n-1})(\alpha_n - \alpha_{n-1})^2}{2!} \\
&\quad -[\phi(t,\alpha_n) - \phi(t,\alpha_{n-1})] + g(t,\beta_{n-1}) + G^{(1)}(t,\beta_{n-1})(\beta_n - \beta_{n-1}) \\
&\quad + \frac{G^{(2)}(t,\alpha_{n-1})(\beta_n - \beta_{n-1})^2}{2!} - [\psi(t,\beta_n) - \psi(t,\beta_{n-1})] \\
B\beta_n(\mu) &= b_\mu.
\end{aligned}
$$

which converges uniformly and monotonically to the unique solution of (3.1) and the convergence is of order 3.

Similar results can be obtained for the other two coupled upper and lower solutions of (3.1) which are given by

$$
\begin{aligned}
&-\alpha_0'' \le f(t,\alpha_0) + g(t,\beta_0), \quad B\alpha_0(\mu) \le b_\mu \quad \text{on } J, \\
&-\beta_0'' \ge f(t,\beta_0) + g(t,\alpha_0), \quad B\beta_0(\mu) \ge b_\mu \quad \text{on } J,
\end{aligned}
\tag{3.36}
$$

and

$$
\begin{aligned}
&-\alpha_0'' \le f(t,\beta_0) + g(t,\beta_0), \quad B\alpha_0(\mu) \le b_\mu \quad \text{on } J, \\
&-\beta_0'' \ge f(t,\alpha_0) + g(t,\alpha_0), \quad B\beta_0(\mu) \ge b_\mu \quad \text{on } J.
\end{aligned}
\tag{3.37}
$$

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