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The V-trace of abelian topological groups; a generalization

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Abstract

Let V be some fixed abelian topological group and a right module. In this paper we study the V-trace ,in the category of topological groups ,which is closely related to the continuous homomorphisms . Also we generalize the V-trace and the construction of V-trace will be continued transfinitely

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Introduction

This paper is a continuation of [6] and [8]. Let F and V be some fixed abelian topological groups and right modules . In [6],we defined two classes of groups ,$T(F)$-group and $E(V)$-group by means of the tensor product and continuous homomorphisms of abelian topological groups,respectively. Also we proved some properties of these classes. In [8]we considered two other classes , F-neutralizer and V-trace ,in the category of topological groups.

All rings are associative , unital and modules are unitary,and groups abelian. We denote the categories of right and left S-modules ,for the ring S, by mod-S and S-mod, respectively; F and V stand for some fixed abelian topological groups and right modules. A *Free topological group* is in Markov sense [3].A topological extension of G is a short exact sequence $0 \to N \to Q \xrightarrow{\pi} G \to 0$. In section 1, we recall the F-neuturalizer [6] . In section 2, the construction of V-trace will be continued transfinitely.

1 F-neuturalizer

For some fixed abelian topological groups and right modules,F and V in [6] , $T(F)$-radical, $E(V)$-radical are defined by means of the tensor product and continuous homomorphisms, respectively. These are The dual notions of each other. For the tensor product of abelian topological groups [see, 7].
Let A be an abelian topological group.

Definition 1.1. An S-module A is a $T(F)$-group if $A \otimes F = 0$. We denote the class of $T(F)$-groups by $\tau(F)$. By [6, Proposition 1.2], $\tau(F)$ is closed under homomorphic image , extension and direct sum.

Let V be an abelian topological group and a right S-module.

Definition 1.2. An S-module A is an $E(V)$-group if $Hom(V, A) = 0$. The class of all $E(V)$-groups is denoted by $\mathcal{E}(V)$.

The class $\mathcal{E}(V)$ is closed under , subgroups , extensions and direct products [6].

Definition 1.3. The category mod-S is endowed with a preradical λ ,if for each S-module A , some subgroup $\lambda(A)$ is assigned so that for every continuous S-homomorphism $\phi : A \to B$, $\phi(\lambda(A)) \subset \lambda(B)$. A λ-*radical* means the class of all S-modules for which $\lambda(A) = A$. The class of all S-modules where $\lambda(A) = 0$ is called λ-*semisimple*. A preradical λ is *idempotent* if $\lambda(\lambda(A)) = A$. For more information on the radical theory [see 4,1] .

Let F be a left S-module. Denote by $W_F(A)$ the sum of all subgroups B of an S-module $A \in mod - S$ such that B is a $T(F)$-group.

By [1,chapter 1,Lemma 1.6 and proposition 2.2] W_F is an idempotent radical and $\tau(F)$ is its radical class. This idempotent radical is called $T(F)$-*radical*. Let V be a right S-module and $H_V(A)$ the intersection of all subgroups B of $A \in mod - S$ such that A/B is an $E(V)$-group. The H_V is an idempotent radical and its semisimple class coincides with $\mathcal{E}(V)$ [1]. This idempotent radical is called $E(V)$-*radical*.

Let R and S be two rings and $e : S \to R$ a ring homomorphism.We regard an R-module A as an S-module by putting $as = ae(s)$ for every $a \in A$ and $s \in S$.It easy to show that R and $e(S)$ are S-S-bimodules. Let $R_0 = R/\overline{e(S)}$. Note that R_0 is an S-S-bimodules [9]. Let $F = V = R_0$. If A is an R-module , consider the canonical epimorphism $h : A \otimes_S R \to A \otimes_R R$ where $h(a \otimes_S r) = a \otimes_R r$,$a \in A,\ r \in R$.

Definition 1.4. If $h : A \otimes_S R \to A \otimes_R R$ is an isomorphism , then an R-module A is called a T-module with respect to the ring homomorphism e. Let τ be the class of all such T-modules.

Definition1.5. Let $W(A)$ be the sum of all subgroups B of A such that $B \in \tau$.

As a result we have:

Lemma 1.6. $W(A) \subset W_F(A)$

Proof. The class τ contains an R-module A iff $A \otimes_S F = 0$ where $F = R_0$ [1]. Hence the elements of τ are R-modules which regarded as attracting S-module ,belong to $\tau(F)$. So

$$\tau = \tau(F) \cap mod - R$$

Therefore by this equality and definitions of $W_F(A)$ and $W(A)$ we have $W(A) \subset W_F(A)$.

We recall from [6] that for an S-module A the F*-neutralizer* of A is the set of all elements a such that $a \otimes f = 0$ for every $f \in F$. We denote by $n_F(A)$ the closure of the normal subgroup generated by the neutralizer elements. Clearly $n_F(A)$ is a normal subgroup of A [5].

The construction of neutralizer can be continued transfinitely [8]. Fix an S-module A . Let $n_F^1(A) = n_F(A)$ If β is a limit ordinal then put $n_F^\beta(A) = \cap_{\alpha \leq \beta}\, n_F^\alpha(A)$. If $\beta = \alpha + 1$ for some ordinal α then put $n_F^\beta(A) = n_F(n_F^\alpha(A))$. Thus we obtain a decreasing chain of submodules $n_F^1(A), n_F^2(A), ..., n_F^\alpha(A), ...$ The chain stabilizes [9] , and we have $n_F^\sigma(A) = n_F^{\sigma+1}(A)$ for some ordinal σ. Notation: $n_F^\infty(A) = n_F^\sigma(A)$.

Now by [8,proposition 2.4],for every ordinal α ,$n_F^\alpha(A)$ is a submodule of the R-module A.

Remark 1.6. Let $A \in$ mod-R. By lemma 1.6, $W(A) \subset W_F(A)$.Also the definition of $W(A)$ shows that $W_F(A) \subset W(A)$. Hence,$W_F(A) = W(A)$.
If $F = R_0$ we omit "F" in "W_F", "T(F)-module" and "T(F)-radical and simply

say "T-module","T-radical",respectively.
Note that every R-homomorphism is automatically an S-homomorphism. So $Hom_R(R, A) \subset Hom_S(R, A)$ for every R-module A.

Definition 1.7. An R-module A is an E-module with respect to the ring homomorphism $e : S \to R$ if $Hom_R(R, A) = Hom_S(R, A)$.

We call such A an "E-module" and denote the class of all E-module over R by ε.

We denote by $H(A)$ the intersection of all subgroups B of A such that $A/B \in \varepsilon$. An R-module A is in ε iff $Hom(V, A) = 0$ when $V = R_0$ [2]. Thus ε consists of those R-modules which belongs to $\varepsilon(V)$ as attracting S-modules i.e. $\varepsilon = \varepsilon(V) \cap$ mod-R . Comparing the definitions of $H_V(A)$ and $H(A)$, $H_V(A) \subset H(A)$.

2 V-trace

In this section, motivated by [9], the construction of V-trace will be continued transfinitely.

A preradical λ is *cotorsion* if

(1) $\lambda(\lambda(A)) = \lambda(A)$ for every $A \in mod - S$

(2) $\lambda(A/B) = \lambda(A) + A/B$ for every $A \in mod - S$ and $B \subset A$

Let V be a right module

Definition 2.1. Let A be an S-module. The *V-trace* of A is the closure of the normal subgroup generated by the sum of ranges of all continuous homomorphisms $\phi \in Hom_S(V, A)$. We denote it by $trace_V(A)$

It is well known that $trace_V$ is an idempotent preradical and the following holds [8]:

$$H_V(A) = 0 \Leftrightarrow A \in \varepsilon(V) \Leftrightarrow Hom_S(V, A) = 0 \Leftrightarrow trace_V(A) = 0$$

Remark 2.2. The definition of V-trace can be reformulated ,as in the case of neutralizer [6] : consider the exact sequence

$$0 \to trace(A) \xrightarrow{\alpha} A \xrightarrow{\beta} A/trace_V(A) \to 0$$

and the induced sequence (not exact in general)

$$0 \to Hom(V, trace(A)) \overset{\alpha_*}{\to} Hom_S(V, A) \overset{\beta_*}{\to} Hom_S(V, A/trace_V(A)) \to 0 \qquad (**)$$

$trace_V(A)$ is the smallest submodule of A with $\beta_* = 0$.
If in $(**)$ $\beta_* = 0$, then $Hom_S(V, A/trace_V(A)) = 0$. Thus $A/trace_V(A) \in \varepsilon(V)$. So $H_V(A) \subseteq trace_V(A)$. Recalling the definition of $E(V)$-radical, by [8] , H_V is the smallest among the idempotent radicals and $trace_V(A) \subseteq H_V(A)$. Under the following condition the equality holds:

Proposition 2.3. *Let V be a projective topological module, Then*

(1) $trac_V(A) = H_V(A)$.

(2) H_V is a cotorsion

Proof.(1). It is enough to show that $H_V(A) \subseteq trace_V(A)$. Since V is projective every sequence

$$0 \to Hom(V, trace(A)) \overset{\alpha_*}{\to} Hom_S(V, A) \overset{\beta_*}{\to} Hom_S(V, A/trace_V(A)) \to 0$$

is exact. Now $\beta_* = 0$, so $Hom_S(V, A/trace_V(A)) = 0$. Thus $A/trace_V(A) \in \varepsilon(V)$ and hence $H_V(A) = trace_V(A)$.

(2). Let $\beta : A \to A/B$ be the natural onto continuous homomorphism. By the above , $\beta_* : Hom_S(V, A) \to Hom_S(V, A/B)$ is onto. So if $Hom_S(V, A) = 0$, then $Hom_S(V, A/B) = 0$. Therefore, for every submodule B of A the condition $A \in \epsilon(A)$ implies $A/B \in \epsilon(A)$. Thus $\epsilon(A)$ is closed under homomorphic images. So H_v is cotorsion.

The construction of V-trace can be continued transfinitely . Let $A \in mod-S$. To each ordinal α , assign the subgroup $trace_V^\alpha(A)$. Put $trace_V^1(A) = trace_V(A)$. If β is a limit ordinal , then $trace_V^\beta(A) = \cup_{\alpha<\beta} trace_V^\alpha(A)$. If $\beta = \alpha + 1$ for some ordinal α , then choose $trace_V^\beta(A)$ so that

$$trace_V(A/trace_V^\alpha(A)) = trace_V^\beta(A)/trace_V^\alpha(A)$$

Now obtain an increasing chain

$$trace_V^!(A), trace_V^2(A), ..., trace_V^\alpha(A)...$$

which obviously must stabilize[9] and so

$$trace_V^\sigma(A) = trace_V^{\sigma+1}(A)$$

for some ordinal σ. Put $trace_V^{\infty}(A)$ $=trace_V^{\sigma}(A)$. By [8,proposition 2.8], $trace_V(A)$ is a submodule of the R-module A. The following generalizes the above result:

Proposition 2.4 *For every ordinal α, $trace_V^{\alpha}(A)$ is a submodule of the R-module A*

Proof. Let $V = R/\overline{e(S)}$, and $\overline{r}$,$\overline{r_1}$... be elements of V.It is enough to show that $(trace_V(A))R \subset trace_V(A)$.
Let$\phi \in Hom_S(V, A)$, $\Phi(\overline{r_1}) \in Im\phi$ and $r_2 \in R$. We define $\Psi : R \to A$ by

$$\Psi(r) = \Phi(\overline{r}) \quad for\ any \ \ r \in R$$

It is clear that ψ is a continuous homomorphism and $\psi \in Hom_S(R, A)$. Define a map $\chi : R \to A$ by

$$\chi(r) = \Psi(r_1)r - \Psi(r_1 r)$$

clearly χ is additive and

$$\chi(rs) = \Psi(r_1)rs - \Psi(r_1 rs) = ((\Psi(r_1))r - \Psi(r_1 r))s = \chi(r)s$$

for any $s \in S$. Hence $\chi \in Hom_S(R, A)$.
For all $s \in S$, $\chi(e(s)) = \Psi(r_1)s - \Psi(r_1 s) = 0$ so $e(s) \in Ker\chi$. Now χ induces a continuous homomorphism $\overline{\chi} \in Hom_S(V, A)$ by $\overline{\chi}(\overline{r}) = \chi(r)$. Since $\chi(r_2) = \Psi(r_1)r_2 - \Psi(r_1 r_2)$, so $\Psi(r_1)r_2 = \chi(r_2) + \Psi(r_1 r_2)$. Now if we consider the homomorphism $\phi : V \to R$, then $\Phi(\overline{r}_1)r_2 = \overline{\chi}(\overline{r}_2) + \Phi(\overline{r_1 r_2})$, that is $\Phi(\overline{r}_1)r_2 \in trace_V(A)$. So $Im\Phi \subset trace_V(A)$ for all ϕ. Hence

$$(trace_V(A))R \subset trace_V(A)$$

Thus $trace_V(A)$ is a submoule of A.

If β is a limit ordinal and $trace_V^{\alpha}(A)$ is a submodule in A for every$\alpha < \beta$ then $trace_V^{\beta}(A)$ is also , by definition, a submodule in A. If $\beta = \alpha + 1$,Then since $trace_V^{\alpha}(A)$ is a submodule of the R-module $A/trace_V^{\alpha}(A)$, it follows that $trace_V^{\beta}(A)$ is also a submodule of A.

Note that in general $trace_V(A) \subseteq H_V(A)$ and by proposition 2.3, when V is a projective topological group, the equality holds. In the case of $trace_V^{\infty}(A)$, we have the following result:

Proposition 2.5 $trace_V^{\infty}(A) = H_V(A)$ *for some S-module A*

Proof. By induction , We show that $trace_V^{\alpha}(A) \subset H_V(A)$. For $\alpha = 1$, this is the above result . Let β be a limit ordinal. Since the inclusion holds for all

$\alpha < \beta$, we have $trace_V^{\beta}(A) \subset H_V(A)$. Now suppose that $\beta = \alpha + 1$ for some α. Since H_V is a radical and by assumption ,

$$H_V(A) = H_V(H_V(A)) \supset H_V(trace_V^{\alpha}(A)) \supset trace_V(trace_V^{\alpha}(A)) = trace_V^{\alpha+1}(A)$$

This completes the induction steps. If $\alpha = \sigma$,then $trace_V^{\infty}(A) = trace_V^{\sigma}(A) \subset H_V(A)$. Furthermore, H_V is the intersection of all B submodule of $A \in mod{-}S$ such that $A/B \in \epsilon(V)$ and $trace_V^{\sigma+1}(A) = trace_V^{\sigma}(A)$. Hence $trace_V^{\sigma}(A) \supset H_V(A)$.

Remark . Like every left S-module F generats an idempotent radical W_F in mod-S, a right S-module V generats an idempotent radical $W_V^{'}$ in S-mod. Hence we can say that a homomorphism $e : S \to R$ generats four radicals: W and H in the category mod-R of right modules and the radicals $W^{'}$ and $H^{'}$ in R-mod.

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