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A NONLINEAR VARIATIONAL PROBLEM OF PROFIT MAXIMIZATION WITH CONVEX COSTS

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Abstract

In this article some optimal production plans are obtained by a deterministic profit maximization, arriving at a inflation scenery description. The industrial (production/storage) costing rate is assumed to be a positive, increasing, convex, fourth degree function of the production itself. By maximizing the integral of profit, in lack of inflation, the variational calculus leads to a Euler-Lagrange nonlinear boundary value problem which we turn to an auxiliary Cauchy nonlinear problem whose solution is proved to be always strictly increasing. Under inflation, introducing further constraints between costs and inflation itself, one is driven to a Duffing equation. In both contexts anyway, optimized production plans within a finite time horizon are amenable to exact treatments and formulated by means of Jacobi elliptic functions.

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1 Introduction

Manufacture planning and control entails the acquisition and allocation of limited resources to production activities so as to satisfy customer demand over a specified time horizon. As such, best planning and control are inherently optimization problems, where the objective is either a plan that meets demand at minimum cost, or that maximizes profit in filling demand. In this paper we will know how a manufacturer could get his (or her) *maximum profit*: he (or she) is assumed to operate deterministically, with different kinds of cost functions, within a finite and prescribed horizon of time. He formulates his production plan by solving nonlinear boundary value problems of Euler-Lagrange equation of variational calculus. But this is not the sole possible approach: deterministic versus stochastic; continuous time versus discrete; variational calculus versus dynamical program, are some possible antagoniste ones; for each alternative our choice is always the first. We will assume a continuous time, finite horizon scenery in which a rational planner is searching his maximum profit, operating within an invariable environment whose parameters connected to demand, storage, revenue and inflation, are independent and invariant during time. A first touch with the problem comes from the Kamien-Schwartz model [4], followed by production optimal planning foundation. At section 3, industrial convex costs are considered, treating the optimization of all quartic/convex cost sceneries without inflation. The solution to a special problem with inflation closes the work. Strictly increasing behaving solutions are obtained- if a certain inequality is met- in lack of inflation. Inflation, at high rates, makes monotonicity undecidable; while for low ones, coming close to the zero rate solution, our Lemma 1.1 about monotonicity is applicable in asymptotic way.

Kamien-Schwartz model and generalizations

The Kamien-Schwartz model, [4], is founded upon a *storage costs minimization.* At time $t = 0$, a firm is required to deliver within time $T > 0$ a quantity $B > 0$ of goods of his production. The firm supports some instantaneous burdens, where U is the cumulative quantity of goods produced from $t = 0$ up to t:

i) cost rate due to production speed: $c\dot{U}^2(t)$, $c > 0$,

ii) cost rate due to goods storage: $\phi_0 + \phi_1 U(t)$, ϕ_0, $\phi_1 > 0$.

If the discount rate $\delta > 0$ is constant in $[0, T]$, the total cost associated to a certain productive plan $U(t)$ will be:

$$\int_0^T e^{-\delta t}\left[c\dot{U}^2(t) + \phi_0 + \phi_1 U(t)\right] dt, \tag{1}$$

with the boundary conditions $U(0) = 0$, $U(T) = B$. The firm needs a plan capable of minimizing the integral (1). The Euler-Lagrange equation associated to (1), leads to the boundary value problem:

$$\begin{cases} \ddot{u}(t) = \delta \dot{u}(t) + \dfrac{\phi_1}{2c}, \\ u(0) = 0,\ u(T) = B > 0. \end{cases}$$

Solving, we find the best productive plan $u(t)$:

$$u(t) = \frac{1}{2\delta c}\Big[\frac{2B\delta c + T\,\phi_1}{e^{T\delta} - 1}\left(e^{t\delta} - 1\right) - \phi_1 t\Big].$$

Observe that $\ddot{u}(t) > 0$ so that $\dot{u}(t)$ is increasing if $\dot{u}(0) \geq 0$, i.e.:

$$B \geq \frac{\phi_1}{2\delta^2 c}\left(e^{\delta T} - 1 - \delta T\right). \tag{2}$$

Inequality (2) guarantees strict growth for the optimal solution, avoiding any productive loss. This model, making use of an affine function for storage costs, allows a full computability of everything. Nevertheless it cannot be deemed satisfactory for all possible productions: accordingly, we are turning to a nonlinear cost model, using quadratic and quartic functions. On the contrary, the kinetic term $c\,\dot{u}^2(t)$, gives the supplementary cost rate due to speed, and, for its own nature, does not require any generalization.

A profitable production optimization

A firm produces and sells a good which can be stored, and acts in continuous time on a finite time span $[0, T]$. A *production plan* is a suitable time law giving the cumulative quantity u of the goods produced *up to time* t, so that a rational agent succeeds in getting the maximum profit associated to $u(t)$ belonging to a suitable class $\mathbb{K}$ of *admissible paths* $\max\limits_{u\in\mathbb{K}} \mathcal{P}(u(t))$. Of course the set $\mathbb{K}$ of admissible paths depends on the nature of the functional $\mathcal{P}$. In our work the natural environment is $\mathbb{K} = H^1([0, T])$, the Sobolev space of the *absolutely continuous functions* with derivative in $L^2([0, T])$ and, from the following assumption on the cost-structure it is a straightforward that solution is regular, see, for example, chapter 4 of [2]. The firm knows the revenue associated to a given level of sales, as well as its production and storage costs. The planner is here interested, not in minimizing costs, but in maximizing his profit, namely the revenue diminished of production/storage costs:

$$\max_{u\in\mathbb{K}} \int_0^T e^{-\delta t}\left[Au - c\,\dot{u}^2(t) - \Phi(u(t))\right] dt,$$

where δ models inflation, Au is the revenue rate, and $c\,\dot{u}^2$ is the cost rate due to the speed of production. Finally, $\Phi(u)$ is the polynomial cost rate for producing and storing goods:

$$\Phi(u) = \sum_{j=1}^{n} \left[\alpha_j^{(p)} + \alpha_j^{(s)}\right] u^j.$$

Joining the effects of production and storing, we put: $\alpha_j^{(p)} + \alpha_j^{(s)} = \alpha_j$, so that our production plan detection becomes: to find $u(t)$ such that:

$$\max_{u\in\mathbb{K}} \mathcal{P}(u(t)) = -\min_{u\in\mathbb{K}} \left\{ \int_0^T e^{-\delta t} \left[\sum_{j=1}^{n} \alpha_j u^j(t) - Au + c\,\dot{u}^2(t) \right] dt \right\},$$

for given α_j, and being assigned the invariant values A, δ, c, $T > 0$. Let $\Psi(u) = \Phi(u) - Au$ be an increasing and convex function: we are then led to find $u(t)$ such that:

$$\min_{u\in\mathbb{K}} \int_0^T e^{-\delta t} \left[c\,\dot{u}^2(t) + \Psi\left(u(t)\right) \right] dt. \tag{3}$$

The model (3) is more general than (1), which can be seen as its first order approximation. The optimum path will be found solving:

$$\begin{cases} \ddot{u}(t) = \delta \dot{u}(t) + \dfrac{1}{2c} \Psi'\left(u(t)\right), \\ u(0) = 0,\ u(T) = B > 0. \end{cases} \tag{4}$$

The model we are about to bring in, has the main following features:

i) takes from [4] the cost structure with both kinetic $c\,\dot{u}^2(t)$ and production dependence, but with nonlinearity effects for the latter and maximizes profit instead of minimizing costs;

ii) assumes the same structure for both industrial costs of production and storage: the cost function $\Psi(u)$ is a polynomial, increasing and convex one;

iii) takes into account the quadratic case: $\Psi = \Psi_2(u) = \phi_0 + \phi_1\, u + \phi_2\, u^2$ and a fourth degree cost rate $\Psi = \Psi_4(u) = \phi_0 + \phi_1\, u + \phi_2\, u^2 + \phi_3\, u^3 + \phi_4\, u^4$ is examinated, considering first $\delta = 0$.

When and why industrial costs are convex

The main characteristic of scale economy is: marginal production cost $\Psi(u)$ decreases as a function of produced units so that there is a reduction of growth's rate of $\Psi'(u)$ as u increases. Costs tend then to "decelerate" and their speed saturates, so that for high u values, $\Psi'(u)$ will approximate a constant, and then $\Psi(u)$ becomes simply proportional to u. What above implies $\Psi''(u) < 0$ and then concavity for costs function, which, for ordinary goods, is likely even considering the storage ones. But the opposite happens if an agent has to manage dangerous, toxic or contaminating substances, such inventories of explosives or radioactive wastes. For them, in order to prevent extreme dangers and/or ecological or biological damnages, higher quantities will require more safety controls, monitoring, additional checks, and special handling; so that increasingly higher, and then accelerating, costs will correspond to a stock rising up. Therefore, the operating cost (production plus storage) rates of "not ordinary" goods, are correctly modelled by increasing and *convex* functions of the production amount, instead of concave ones. From the algebraic point of view, global convexity of $\Psi(u) = \phi_0 u^0 + \cdots + \phi_n u^n$, prescribes n even, and we will concentrate on degrees $n = 2$ and $n = 4$. Finally, we note that the convexity requirement makes our minimization problem *well posed*.

A useful Lemma

After the firm production cost structure has been characterized, let us give a lemma concerning the optimization problem (3) and the relevant Euler Lagrange (4) for the scenery $\delta = 0$.

Lemma 1.1. *Let $\Psi \in C^2(\mathbb{R}), \Psi'(\xi), \Psi''(\xi) > 0$ a nonnegative function such that $\lim_{\xi\to\infty} \Psi(\xi) = \infty$. The optimum problem:*

$$\inf_{u\in\mathbb{K}} \int_0^T \left[c\dot{u}^2(t) + \Psi(u(t))\right] dt,\ u(0) = 0,\ u(T) = B > 0,\ c > 0,$$

admits a unique regular and strictly increasing solution $u(t)$ if:

$$T \le \sqrt{c}\int_0^B \frac{du}{\sqrt{\Psi(u) - \Psi(0)}}. \tag{5}$$

Proof. The existence and regularity of the solution are well known topics. First notice that integral in right hand side of (5) converges. The Euler Lagrange equation is

$$\begin{cases} \ddot{u}(t) = \dfrac{1}{2c}\Psi'(u(t)), \\ u(0) = 0,\ u(T) = B. \end{cases} \tag{6}$$

Consider now the initial value problem associated to (6):

$$\begin{cases} \ddot{u}(t) = \dfrac{1}{2c}\Psi'\left(u(t)\right), \\ u(0) = 0, \dot{u}(0) = x > 0. \end{cases} \tag{7}$$

Differential equation (7) is integrated by Weierstraß method, [1], and by direct inspection we see that solution of (7) also solves (6) if it is possible to find a speed $x > 0$ such that:

$$T = \sqrt{c}\int_0^B \frac{du}{\sqrt{\Psi(u) - \Psi(0) + cx^2}}.$$

On the other side:

$$x \mapsto \sqrt{c}\int_0^B \frac{du}{\sqrt{\Psi(u) - \Psi(0) + cx^2}},$$

is a strictly decreasing function, then thesis follows from (5). □

By the Weierstraß method we will solve (4) in closed form, getting solutions which can be strictly increasing, what is compliant with the model economic soundness, as ensuring inventory development without any loss. Finally, let us note (2) to be nothing else than (5) whenever $\delta \to 0$.

Quadratic costs with inflation

The optimization problem in such a case becomes:

$$\begin{cases} \min\limits_{u\in\mathbb{K}} \displaystyle\int_0^T e^{-\delta t}\left[c\,\dot{u}^2(t) + \phi_0 + \phi_1 u(t) + \phi_2 u^2(t)\right] dt, \\ u(0) = 0,\ u(T) = B > 0, \end{cases} \tag{8}$$

the strictly convexity and monotonicity for $u > 0$, requires:

$$\phi_0 > 0,\ \phi_1 > 0,\ \phi_2 > 0,\ \phi_1^2 - 4\phi_0\phi_2 < 0. \tag{9}$$

We are led to an elementary Dirichlet boundary value problem:

$$\begin{cases} \ddot{u}(t) = \delta\dot{u}(t) + \dfrac{\phi_2}{c}u(t) + \dfrac{\phi_1}{2c}, \\ u(0) = 0,\ u(T) = B, \end{cases}$$

whose solution holds hyperbolic functions:

$$u(t) = Be^{\delta(t-T)/2}\,\mathrm{csch}\,(RT)\,\sinh\,(Rt) - \frac{\phi_1}{2\phi_2} + \frac{\phi_1}{2\phi_2}\times$$
$$\times\Big\{\left[\cosh\,(Rt) - \coth\,(RT)\,\sinh\,(Rt)\right]e^{\delta t/2} + \mathrm{csch}\,(R\,T)\,\sinh\,(R\,t)\,e^{\delta(t-T)/2}\Big\},$$

where $R = \sqrt{\dfrac{c\,\delta^2 + 4\phi_2}{4c}}$. If $\delta = 0$, for Lemma 1.1, monotonicity condition is:

$$T \le \sqrt{\frac{c}{\phi_2}}\,\text{arccosh}\frac{2B\phi_2 + \phi_1}{\phi_1}.$$

2 Optimal plan: quartic costs, no inflation

We are now searching the optimal production plan in an idealized macroeconomic scenery where inflation does not occur, doing a closed form integration for each possible case. Our treatment is divided in two steps. First, integration is carried out in closed form. Afterwards, we will look for *inverting* that, namely to obtain for u as a function of time.

The model set-up

The profit maximization problem with fourth degree costs is:

$$\begin{cases} \min \displaystyle\int_0^T e^{-\delta t}\left[c\,\dot u^2(t) + \phi_0 + \phi_1 u(t) + \phi_2 u^2(t) + \phi_3 u^3(t) + \phi_4 u^4(t)\right] dt, \\ u(0) = 0,\ u(T) = B > 0. \end{cases}$$

Presently we restrict to $\delta = 0$: inflation will be faced next section. For $u > 0$, we assume inequalities:

$$\phi_1 > 0,\ \phi_2 > 0,\ \phi_4 > 0,\ \phi_3^2 < \frac{8}{3}\,\phi_2\phi_4, \tag{10}$$

to hold in order to ensure convexity and monotonicity for $u > 0$. The search of the production law $u(t)$ capable of getting the above minimum, will ensure the best profit to the agent. As algebraic consequence, the fourth degree polynomials suitable to model industrial costs will be of three kinds only:

i) no real roots;

ii) two negative real roots;

iii) one negative double real root.

The optimality condition is:

$$\begin{cases} \ddot u(t) = \delta \dot u(t) + \dfrac{1}{2c}\left[\phi_1 + 2\phi_2 u(t) + 3\phi_3 u^2(t) + 4\phi_4 u^3(t)\right], \\ u(0) = 0,\ u(T) = B > 0. \end{cases} \tag{11}$$

Let $\delta = 0$. So from Lemma 1.1 we get the monotonicity condition:

$$T \le \sqrt{c}\int_0^B \frac{du}{\sqrt{\phi_1 u + \phi_2 u^2 + \phi_3 u^3 + \phi_4 u^4}}. \tag{12}$$

From [3], integral 260.00 page 135, (12) can be written as:

$$T \leq \frac{\sqrt{c}}{\phi_4} \frac{F(\varphi_0, k_0)}{\sqrt[4]{q(r^2 - pr + q)}}, \tag{13}$$

where $F(\varphi, k)$ is the normal incomplete elliptic integral of first kind with amplitude φ and modulus k, see [3], formula 110.02 page 8 and:

$$\begin{aligned}
\varphi_0 &= \arccos\left[\frac{B\left(\sqrt{q} - \sqrt{r^2 - pr + q}\right) + r\sqrt{q}}{B\left(\sqrt{q} + \sqrt{r^2 - pr + q}\right) + r\sqrt{q}}\right], \\
k^2 &= \frac{2q - pr + 2\sqrt{q(r^2 - pr + q)}}{4\sqrt{q(r^2 - pr + q)}}, \\
\phi_1\xi + \phi_2\xi^2 + &\ \phi_3\xi^3 + \phi_4\xi^4 = \phi_4\,\xi\,(\xi + r)\left(\xi^2 + p\xi + q\right),
\end{aligned}$$

being last equation due to assumptions (10) for suitable p, q, r choices.

Integration

Our goal is to carry out:

$$t = \sqrt{c}\int_0^u \frac{d\xi}{\sqrt{cx^2 + \phi_1\xi + \phi_2\xi^2 + \phi_3\xi^3 + \phi_4\xi^4}}, \tag{14}$$

in any of its admittable occurrences which depend on the initial speed $x = \dot{u}(0)$. Define $P_x(\xi) = cx^2 + \phi_1\xi + \phi_2\xi^2 + \phi_3\xi^3 + \phi_4\xi^4$ so that for (10) $P_x(\xi)$ has an unique minimum. In such a way cubic equation $P'(\xi) = 0$ has one real negative root, say ξ_r , which can be explicitly computed solving the relevant cubic equation, but this does not give any further insight on the problem. The sign of $P_x(\xi_r)$ rules integration. As a matter of fact:

a) $P_x(\xi_r) > 0$, no real roots for $P_x(\xi)$: hence for some a_1, a_2, b_1, $b_2 \in \mathbb{R}$:

$$P_x(\xi) = \phi_4\left((\xi - b_1)^2 + a_1^2\right)\left((\xi - b_2)^2 + a_2^2\right).$$

b) $P_x(\xi_r) < 0$, two negative roots for $P_x(\xi)$: for some a, b, a_1, $b_1 \in \mathbb{R}$, a, $b < 0$, $b < a$:

$$P_x(\xi) = \phi_4(\xi - a)(\xi - b)\left((\xi - b_1)^2 + a_1^2\right).$$

c) $P_x(\xi_r) = 0$, one double real root for $P_x(\xi)$: for some a, a_1, $b_1 \in \mathbb{R}$, $a < 0$:

$$P_x(\xi) = \phi_4(\xi - a)^2\left((\xi - b_1)^2 + a_1^2\right).$$

Let us integrate (14). In case a), use formula 267.00, page 146, [3]. Putting:

$$\begin{aligned} A^2 &= (b_1-b_2)^2+(a_1+a_2)^2, \ \ B^2=(b_1-b_2)^2+(a_1-a_2)^2; \\ k^2 &= \frac{4AB}{(A+B)^2}, \ g=\frac{2}{A+B}, \ g_1^2=\frac{4a_1^2-(A-B)^2}{(A+B)^2-4a_1^2}, \\ y_1 &= b_1-a_1g_1, \ \varphi(y)=\arctan\frac{y-b_1+a_1g_1}{a_1+g_1b_1-g_1y}; \end{aligned}$$

we have:

$$\int_{y_1}^{y}\frac{d\xi}{\sqrt{\left((\xi-b_1)^2+a_1^2\right)\left((\xi-b_2)^2+a_2^2\right)}}=gF\left(\varphi(y),k\right), \qquad (15)$$

so that (14), becomes $t=\frac{\sqrt{c}}{\sqrt{\phi_4}}g\left[F\left(\varphi(u),k\right)-F\left(\varphi(0),k\right)\right].$

In case b), use formula 260.00, page 135, [3]. Putting:

$$\begin{aligned} A^2 &= (a-b_1)^2+a_1^2, \ B^2=(b-b_1)^2+a_1^2, \ k^2=\frac{(A+B)^2-(a-b)^2}{4AB}; \\ g &= \frac{1}{\sqrt{AB}}, \ \varphi(y)=\arccos\frac{(A-b)\,y+aB-bA}{(A+B)\,y-aB-bA}; \end{aligned}$$

we have:

$$\int_{a}^{y}\frac{d\xi}{\sqrt{(\xi-a)(\xi-b)\left((\xi-b_1)^2+a_1^2\right)}}=gF\left(\varphi(y),k\right),$$

so that (14), becomes: $t=\frac{\sqrt{c}}{\sqrt{\phi_4}}g\left[F\left(\varphi(u),k\right)-F\left(\varphi(0),k\right)\right]$, where the same symbols g, φ, k, as a) have been used with different meanings.

In case c) integration is elementary:

$$t=\frac{\sqrt{c}}{\sqrt{a_1^2+(b_1-a)^2}}\ln\frac{X(u)}{X(0)},$$

where:

$$X(u)=\frac{(b_1-a)\ (b_1-u)+\left(a_1-\sqrt{a_1^2+(b_1-a)^2}\right)\left(a_1+\sqrt{a_1^2+(b_1-u)^2}\right)}{(b_1-a)\,(b_1-u)+\left(a_1+\sqrt{a_1^2+(\ b_1-u)^2}\right)\left(a_1+\sqrt{a_1^2+(b_1-u)^2}\right)}.$$

Inversion: production versus time

Is it possible to invert and find u in case a) and b). In case a) we find:

$$u(t)=\frac{(a_1g_1-b_1)-(a_1+g_1b_1)\ \mathrm{tn}\left(F\left(\arctan\frac{a_1g_1-b_1}{a_1+g_1b_1},k\right)+\left(\frac{1}{g}\sqrt{\frac{\phi_4}{c}}\right)t,k\right)}{g_1\,\mathrm{tn}\left(F\left(\arctan\frac{a_1g_1-b_1}{a_1+g_1b_1},k\right)+\left(\frac{1}{g}\sqrt{\frac{\phi_4}{c}}\right)t,k\right)-1},$$

where $t\in[0,T]$ and $\mathrm{tn}(v,k)$ is the Jacobian elliptic tangent.

In case b) we obtain the optimal production plan:

$$u(t)=\frac{(aB-bA)+(aB+bA)\ \mathrm{cn}\left[F\left(\arccos\frac{a_1g_1-b_1}{a_1+g_1b_1},k\right)+\left(\frac{1}{g}\sqrt{\frac{\phi_4}{c}}\right)t,k\right]}{(b-A)+(B+A)\ \ \mathrm{cn}\left[F\left(\arccos\frac{a_1g_1-b_1}{a_1+g_1b_1},k\right)+\left(\frac{1}{g}\sqrt{\frac{\phi_4}{c}}\right)t,k\right]},$$

where $\mathrm{cn}(v,k)$ is the Jacobian elliptic cosine.

In case c) the solution $t=t(u)$ does not require any special function, but cannot be inverted for u.

3 Optimal plan: quartic costs with inflation: an integrable case

The previously developed treatment is concerning a production-cost micro environment, with a quartic, increasing and convex cost function of the product $u>0$, embedded in a ideal macro-economic context without inflation. Wishing to include inflation in our analysis, we are faced with a more difficult problem being the relevant differential equation not a Weierstraß one: we shall then operate some useful *ansatz*, discarding the preliminary exam of the monotonicity of solution, because in such a more complex context this is rather a computational problem specific of any given case. A customary approach for getting rid of quadratic term from (11), is the Tschirnhaus transformation $u=\eta-(3\,\phi_3)/(4\,\phi_4)$. In such a way we obtain:

$$\begin{cases}\ddot{\eta}=\delta\dot{\eta}+\nu+\alpha\eta+\dfrac{2\phi_4}{c}\eta^3,\\ \eta(0)=\dfrac{3\,\phi_3}{4\,\phi_4},\ \eta(T)=B+\dfrac{3\,\phi_3}{4\,\phi_4},\end{cases}\tag{16}$$

where:

$$\nu:=\frac{\phi_3^3-4\,\phi_2\,\phi_3\,\phi_4+8\,\phi_1\,\phi_4^2}{16c\,\phi_4^2},\ \alpha:=\frac{8\,\phi_2\phi_4-3\,\phi_3^2}{8c\,\phi_4}.\tag{17}$$

The result is a Duffing differential equation, generally not integrable.

Preliminary discussion

As we did for no-inflation case, we will turn again our originary boundary value problem to an equivalent Cauchy one, namely:

$$\begin{cases} \ddot{\eta} = \delta \dot{\eta} + \nu + \alpha \eta + \dfrac{2\phi_4}{c} \eta^3 \\ \eta(0) = \dfrac{3\,\phi_3}{4\,\phi_4},\ \dot{\eta}(0) = x, \end{cases} \tag{18}$$

where the unknown x will be detected asking the x-marked Cauchy solution to meet the *shooting condition*:

$$\eta(T) = B + \frac{3\,\phi_3}{4\,\phi_4}. \tag{19}$$

Now let us try a further change of variables $\eta = \xi\, q(\xi)$, $t = (3 \ln \xi)/\delta$. Accordingly, problem (18) is changed to:

$$\begin{cases} q''(\xi) = \dfrac{9}{\delta^2\, \xi^3} \left[\nu + \left(\alpha + \dfrac{2}{9}\, \delta^2 \right) \xi\, q(\xi) + \dfrac{2\phi_4}{c} \xi^3\, q^3(\xi) \right], \\ q(1) = \dfrac{3\,\phi_3}{4\,\phi_4},\ q'(1) = \dfrac{3}{\delta}\, x - \dfrac{3\,\phi_3}{4\,\phi_4}, \end{cases} \tag{20}$$

where $'$ now means derivative with respect to ξ. After above transformation, we do not have in (20) the first derivative anymore, even if inflation persists. Like the viscous damping acts as a contrast to the motion of a material point, in the same way inflation brakes the wealth's development as requiring a supplementary effort of production. The resolvent for x comes from (19):

$$e^{\delta T/3}\, q(x; e^{\delta T/3}) = B + \frac{3\,\phi_3}{4\,\phi_4}. \tag{21}$$

Notice that, because of the fundamental condition (10), we know *a priori* that (21) admits a solution.

Two cascade assumptions

Our treatment for $\delta > 0$ has been up to now *general and exact*; but due to a more rich content of our model, equation (20) cannot be integrated in closed form, and then, for keeping exact the treatment, the price of some restriction has to be paid. An immediate specialization comes out assuming $\alpha + (2/9)\, \delta^2 = 0$ as a relationship linking rates of cost and inflation. Namely, recalling (17), we can select a triple (ϕ_2, ϕ_3, ϕ_4) for the cost coefficients so that:

$$\frac{8\,\phi_2\phi_4 - 3\,\phi_3^2}{8c\,\phi_4} = \frac{2}{9}\, \delta^2. \tag{22}$$

Condition (22) requires, due to positivity of c and ϕ_4, that $8\,\phi_2\phi_4 - 3\,\phi_3^2 > 0$ which does not bring any news at all: it coincides with the third of (10). Let us now introduce a further and last restriction. If the quantity ν vanishes, namely:

$$\phi_3^3 - 4\,\phi_2\,\phi_3\,\phi_4 + 8\,\phi_1\,\phi_4^2 = 0, \tag{23}$$

then (20) reduces to:

$$\begin{cases} q''(\xi) = \dfrac{18\,\phi_4}{c\,\delta^2}\,q^3(\xi), \\ q(1) = \dfrac{3\,\phi_3}{4\,\phi_4},\ q'(1) = \dfrac{3}{\delta}x - \dfrac{3\,\phi_3}{4\,\phi_4}. \end{cases} \tag{24}$$

Up to this point, it is crucial to analyze the compatibility of (23) with respect to (10) and (22). First write (23) as:

$$\phi_3\left(\phi_3^2 - 4\,\phi_2\,\phi_4\right) + 8\,\phi_1\,\phi_4^2 = 0. \tag{25}$$

Then, recalling (10), we infer that the factor $\phi_3^2 - 4\,\phi_2\,\phi_4$ in (25) is negative: this force $\phi_3 > 0$, that is a severe restriction stemming from (10), but admissible. Summarizing: joining (10), (23) and (22), we obtain a non empty set, namely a hypersurface $\mathcal{H} \subset \mathbb{R}^4$ defined by: $\mathcal{H} = \mathcal{I} \cap \mathcal{J} \cap \mathcal{K}$, where:

$$\begin{aligned} \mathcal{I} &= \left\{(\phi_1, \phi_2, \phi_3, \phi_4) \in \mathbb{R}^4 :\ \phi_1 > 0,\ \phi_2 > 0, \phi_4 > 0, \phi_3^2 < \frac{8}{3}\,\phi_2\phi_4\right\}, \\ \mathcal{J} &= \left\{(\phi_1, \phi_2, \phi_3, \phi_4) \in \mathbb{R}^4 :\ \frac{8\,\phi_2\phi_4 - 3\,\phi_3^2}{8c\,\phi_4} = \frac{2}{9}\,\delta^2\right\}, \\ \mathcal{K} &= \left\{(\phi_1, \phi_2, \phi_3, \phi_4) \in \mathbb{R}^4 :\ \phi_3 > 0\ \&\ \phi_3^3 - 4\,\phi_2\,\phi_3\,\phi_4 + 8\,\phi_1\,\phi_4^2 = 0\right\}. \end{aligned}$$

In such a way for any $(\phi_1,\ \phi_2,\ \phi_3,\ \phi_4) \in \mathcal{H}$ the initial value problem (20) is reduced to (24), which is a Weierstraß one, see again [1].

Detection of $q(\xi)$

For shortness, let us write (24) as:

$$\begin{cases} q''(\xi) = 2A^4\,q^3(\xi), \\ q(1) = q_1 > 0,\ q'(1) = r_1. \end{cases} \tag{26}$$

Integrating we get:

$$\xi - 1 = \frac{\text{sign}\ (r_1)}{A^2}\int_{q_1}^{q(\xi)} \frac{ds}{\sqrt{s^4 + Q(x)}}, \tag{27}$$

where $A^4 = \dfrac{9\phi_4}{c\delta^2}$, $r_1 = 3\left(\dfrac{x}{\delta} - \dfrac{\phi_3}{4\,\phi_4}\right)$, $q_1 = \dfrac{3\phi_3}{4\,\phi_4}$ and:

$$Q(x) := \frac{r_1^2}{A^4} - q_1^4 = \frac{c}{\phi_4}\,x^2 - \frac{c\delta\,\phi_3}{2\,\phi_4^2}\,x + \frac{\phi_3^2}{16\,\phi_4^3}\left(c\delta^2 - \frac{81\,\phi_3^2}{16\,\phi_4}\right).$$

The $Q(x)$ sign will affect integration of (27). $Q(x)$ discriminant is positive, so we have two real distinct roots:

$$x_{1,2} = \frac{\delta\,\phi_3}{4\,\phi_4} \mp \frac{9}{16}\frac{\phi_3^2}{\sqrt{c}\,\phi_4^{3/2}}.$$

Notice that $x_2 > 0$ for any possible occurrence of all parameters, while $x_1 \geq 0$ iff $4\delta\sqrt{c\,\phi_4} \geq 9\phi_3$. Of course:

$$x < x_1 \vee x > x_2 \iff Q(x) > 0, \tag{28}$$
$$x_1 < x < x_2 \iff Q(x) < 0, \tag{29}$$
$$x = x_1 \wedge x = x_2 \iff Q(x) = 0. \tag{30}$$

First Case: $Q(x) > 0$. Fixing the $Q(x)$ sign, integral function is defined for whichever value of x in the relevant interval. But furthermore, a successive discussion shall be set up for the sign of r_1 too. Accordingly, observe that:

$$x > x_2 \Rightarrow r_1 > \frac{3}{\delta}x_2 - \frac{3\,\phi_3}{4\,\phi_4} = \frac{27\,\phi_3^2}{16\delta\,\phi_4^{3/2}\sqrt{c}} > 0,$$

$$x < x_1 \Rightarrow r_1 < \frac{3}{\delta}x_1 - \frac{3\,\phi_3}{4\,\phi_4} = -\frac{27\,\phi_3^2}{16\delta\,\phi_4^{3/2}\sqrt{c}} < 0.$$

The integral in the right hand side of (27) can be evaluated using a formula, proved in Lemma 2.1 of [5]:

$$\int_0^q \frac{ds}{\sqrt{s^4+1}} = \frac{1}{2}F\left(\arccos\frac{1-q^2}{1+q^2}, \frac{1}{\sqrt{2}}\right),$$

which allows to infer the further integral formula, which holds for $H > 0$:

$$\int_p^q \frac{ds}{\sqrt{s^4+H^4}} = \frac{1}{2H}\left[F\left(\arccos\frac{H^2-q^2}{H^2+q^2}, \frac{1}{\sqrt{2}}\right) - F\left(\arccos\frac{H^2-p^2}{H^2+p^2}, \frac{1}{\sqrt{2}}\right)\right].$$

If, for instance, $x > x_2$, we find:

$$\xi - 1 = \frac{1}{2A^2\sqrt[4]{Q(x)}}\left[F\left(\arccos\frac{\sqrt{Q(x)} - q^2(\xi)}{\sqrt{Q(x)} + q^2(\xi)}\right) - F_1(x)\right], \tag{31}$$

where we omitted the modulus $1/\sqrt{2}$ and we set:

$$F_1(x) = F\left(\arccos\frac{\sqrt{Q(x)} - q_1^2}{\sqrt{Q(x)} + q_1^2}\right).$$

Solving (31) for $q(\xi)$, we find the solution to (26):

$$q(\xi) = \sqrt[4]{Q(x)}\sqrt{\frac{1 - \operatorname{cn}\left(F_1 + 2(\xi - 1)A^2\sqrt[4]{Q(x)}\right)}{1 - \operatorname{cn}\left(F_1 + 2(\xi - 1)A^2\sqrt[4]{Q(x)}\right)}},$$

where the module has been again omitted. If $x < x_1$ we arrive likewise at:

$$q(\xi) = \sqrt[4]{Q(x)}\sqrt{\frac{1 - \operatorname{cn}\left(F_1(x) + 2(1 - \xi)A^2\sqrt[4]{Q(x)}\right)}{1 - \operatorname{cn}\left(F_1(x) + 2(1 - \xi)A^2\sqrt[4]{Q(x)}\right)}},$$

as solution to (26). In both cases the relevant equation (21) for x, leading to solve (16), has to be treated numerically.

Second case: $Q(x) < 0$**.** If (29) holds, the initial "q-speed" r_1 which in the previous case was *a priori* unbounded, is now bounded, being:

$$|r_1| < \frac{27\,\phi_3^2}{16\delta\,\phi_4^{3/2}\sqrt{c}}.$$

We have to discriminate again two occurrences: $r_1 \neq 0$ or $r_1 = 0$. In the first case we introduce $\sigma(x) := \sqrt[4]{-Q(x)}$, so that:

$$\xi - 1 = \frac{\operatorname{sign}\ (r_1)}{\sigma(x)\,A^2}\int_{q_1/\sigma(x)}^{q(\xi)/\sigma(x)}\frac{ds}{\sqrt{s^4 - 1}}. \tag{32}$$

Notice that (29) implies $q_1 \geq \sigma(x)$: the problem is well posed. Using once more [3], formula 260.75 page 138, one gets:

$$\xi - 1 = \frac{\operatorname{sign}\ (r_1)}{\sigma(x)\,A^2\sqrt{2}}\left[F\left(\arccos\left(\frac{\sigma(x)}{q(\xi)}\right)\right) - F\left(\arccos\left(\frac{\sigma(x)}{q_1}\right)\right)\right], \tag{33}$$

where modulus $1/\sqrt{2}$ is omitted. Solving (32) for $q(\xi)$:

$$q(\xi) = \sigma(x)\operatorname{nc}\left(F\left(\arccos\left(\frac{\sigma(x)}{q_1}\right)\right) + \operatorname{sign}(r_1)A^2\sqrt{2}\,(\xi - 1)\,\sigma(x)\right),$$

where the Glaisher's notation, see [3] formula 120.02 page 19, is used to designate the reciprocal of cosine amplitude cn. Eventually, when $r_1 = 0$, i.e. when $x = (\delta\phi_3)/(3\phi_4)$, we have:

$$\xi - 1 = \frac{1}{A^2 q_1} \int_1^{q(\xi)/q_1} \frac{ds}{\sqrt{s^4 - 1}},$$

invoking again formula 260.75 page 138 of [3], after some handling we obtain:

$$q(\xi) = q_1 \,\mathrm{nc}\left(A^2\sqrt{2}\,(\xi - 1)\,, \frac{1}{\sqrt{2}} \right).$$

Third case: $Q(x) = 0$**.** If (30) holds, we have two trivial cases. If $x = x_2$, then $r_1 > 0$ so that:

$$q(\xi) = \frac{q_1}{1 - A^2 q_1\,(\xi - 1)}.$$

Inserting the values of q_1 and A, we see that (21) becomes:

$$\frac{-3\sqrt{c}\,e^{\frac{T\delta}{3}}\,\delta\,\phi_3}{\left(9\left(e^{\frac{T\delta}{3}} - 1\right)\phi_3 - 4\sqrt{c}\,\delta\,\sqrt{\phi_4}\right)\sqrt{\phi_4}} = B + \frac{3\phi_3}{4\phi_4},$$

which is a further condition to be imposed on the parameters in order to solve the optimum problem (16) in this particular case. Vice versa, if $x = x_2$, then $r_1 < 0$ so that, after some straightforward calculations, (21) becomes:

$$\frac{3\sqrt{c}\,e^{\frac{T\delta}{3}}\,\delta\,\phi_3}{\left(9\left(e^{\frac{T\delta}{3}} - 1\right)\phi_3 + 4\sqrt{c}\,\delta\,\sqrt{\phi_4}\right)\sqrt{\phi_4}} = B + \frac{3\phi_3}{4\phi_4}.$$

Back to optimal production plan

Whenever one assumes the proper $q(x;\xi)$ function, he or she has to solve *numerically* for x each of three possible equations (21), because it is *a priori* unknown which, among solutions (31), (32), and so on, is fit to our data problem. Nevertheless the Cauchy problem in $q(\xi)$ has an unique solution; and as a consequence, our algebraic problem in x will admit only one x - value. The steps for singling it, and then the relevant q - shape, will be:

- to solve numerically three $x-$problems;
- to pass to $u(t) = \xi(t)q(\xi(t)) - \dfrac{3\,\phi_3}{4\,\phi_4}$, where $\xi(t) = e^{\delta t/3}$;
- to check which solution meets the final condition $u(T) = B$;
- finally, to go back to decide the optimal planning $u(t)$, according to which of the proposed functions matches the correct B value.

4 Conclusions

We computed optimal production plans by maximizing in deterministic way a profit functional within a given T horizon of time. The production-storage cost model intensity has been assumed to be a positive, increasing, convex function of the production amount u. Two contexts have been modelled.

In absence of inflation, a general result, Lemma 1.1 has been proved in order to define, by means of a condition on T, when a solution is strictly increasing so as to avoid any loss. After evaluating the startup inventory growth rate x, everything depends on the unique real root $\xi_r < 0$ of a certain cubic equation, or, better, on the sign of $P_x(\xi_r)$ where P_x is the positive quartic model of the costs parametrized with x. Accordingly, integration can be always carried out exactly, providing the optimal production plan in the course of time. Furthermore, for $P_x(\xi_r) > 0$, or $P_x(\xi_r) < 0$, the relationship between u and t can also be inverted, giving through Jacobi elliptic functions a time-plan whose arguments depend *linearly* upon time. The transition case $P_x(\xi_r) = 0$ can be solved without higher transcendental functions, but cannot be inverted.

In presence of inflation the analytical problem is more difficult, and the monotonicity undecidable. Removing properly quadratic term therefrom, the relevant differential equation is transformed into a damped Duffing equation, so that some restriction has to be added, in order to obtain a closed form solution. Assuming two relationships (22) and (23) among costs and inflation, one arrives at a tractable differential equation. After startup inventory growth rate x has been evaluated, our best profit production plan is displayed in terms of elliptic functions again, but whose argument is a nonlinear function of time. Finally, by comparing the optimal plans under inflation, we see in both of them an *external* exponential multiplier $e^{\delta t}$. In the quartic case, even if modified as $(e^{\delta/3t} - 1)\delta^{-1}$, such a multiplier acts also *inside* the elliptic argument. This means that at any moment t of $[0, T]$, inflation prescribes a production greater than in its absence: in fact profit maximization intrinsicly requires a money compensation for the goods' monetary value lost by inflation between the start off and t.

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