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On the Regularized Trace of a Fourth Order Regular Differential Equation

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Abstract

We shall obtain a formula for the regularized trace of a fourth order regular differential equation.

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1 STATEMENT OF THE PROBLEM

In the space $L_2[0,\pi]$ we consider the self-adjoint operators L_0 and L which are generated by the following expressions:

$$\ell_0(y) = y^{(4)}, \qquad \ell(y) = y^{(4)} + p(x)y$$

with the same boundary conditions

$$y^{(\nu)}(0) = y^{(\nu)}(\pi) \qquad\qquad (\nu = 0,\ 1,\ 2,\ 3)$$

respectively. Here $p(x)$ is a real valued, continuous function in $[0,\ \pi]$.

The spectrum of operator L_0 coincides with the set $\left\{16n^4\right\}_{n=0}^{\infty}$. Every point of the spectrum is an eigenvalue with multiplicity two except point zero. Zero is the simple eigenvalue. We denote the eigenvalues of operator L_0 by $\left\{\mu_k\right\}_{k=0}^{\infty}$

$$\mu_0 = 0 \qquad \text{and} \qquad \mu_k = \begin{cases} (k+1)^4 & \text{if } k \text{ is odd and } k \geq 1 \\ \\ k^4 & \text{if } k \text{ is even and } k \geq 2 \end{cases}$$

Orthonormal eigenfunctions corresponding to this eigenvalues are denoted by

$$\psi_0 = \sqrt{\pi^{-1}}, \quad \psi_1 = \sqrt{2\pi^{-1}}\sin 2x, \quad \psi_2 = \sqrt{2\pi^{-1}}\cos 2x, \cdots$$

We denote the eigenvalues of operator L by $\lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \cdots$ and corresponding orthonormal eigenfunctions by $\varphi_0,\ \varphi_1,\ \varphi_2,\ \varphi_3,\ \cdots$.

In this paper, by Dikiy's method, we will obtain a formula for the sum of series

$$\sum_{n=0}^{\infty}(\lambda_n - \mu_n)$$

which is called regularized trace of operator L.

Firstly the regularized trace formula for the Sturm-Liouville operator have been found by Gelfand-Levitan [1]. The some regularized trace formula for the same problem was obtained with different method by Dikiy [2]. Later study of regularized trace was generalized for different differential operators (see, for example [3]-[8])

2 SOME ESTIMATES.

In this section we prove the formula

$$\lim_{N\to\infty}\sum_{n=0}^{N}\Big[(\varphi_n,\ L\varphi_n) - (\psi_n,\ L\psi_n)\Big] = 0 \tag{2.1}$$

which will be used later. For this purpose we investigate the transfer matrix $(u_{i\,k})_{i,\,k=0}^{\infty}$ from the orthonormal basis $\{\varphi_k\}$ to orthonormal basis $\{\psi_k\}$ as in [2]:

$$\psi_k = \sum_{i=0}^{\infty} u_{i\,k}\varphi_i\,, \qquad \text{where } u_{i\,k} = (\varphi_i,\ \psi_k)$$

$(u_{i\,k})_{i,\,k=0}^{\infty}$ is the a unitary matrix, that is

$$\sum_{k=0}^{\infty} u_{i\,k}^2 = \sum_{i=0}^{\infty} u_{i\,k}^2 = 1.$$

First of all, let us give some estimate for numbers $u_{i\,k}$.
It is clear that

$$L\psi_k = \mu_k \psi_k + p\psi_k \tag{2.2}$$

Scalar producting both side of equality (2.2) by φ_i we obtain

$$(L\psi_k,\ \varphi_i) = (\mu_k\psi_k,\ \varphi_i) + (p\psi_k,\ \varphi_i)$$

or

$$\lambda_i(\psi_k,\ \varphi_i) = \mu_k(\psi_k,\ \varphi_i) + (p\psi_k,\ \varphi_i)$$

and

$$(\lambda_i - \mu_k)(\psi_k,\ \varphi_i) = (p\psi_k,\ \varphi_i)$$

According to [2] taking the square of both sides of the last equality and summing according to i we obtain

$$\begin{aligned}\sum_{i=0}^{\infty}(\lambda_i - \mu_k)^2(\psi_k,\ \varphi_i)^2 = \sum_{i=0}^{\infty}(p\psi_k,\ \varphi_i) &= \|p\psi_k\|^2 = \int_0^{\pi}[p(x)\psi_k(x)]^2 dx \\ &\le p_0^2 \int_0^{\pi} \psi_k^2(x)dx = p_0^2\end{aligned}$$

where $p_0 = \max\limits_{0\le x\le\pi} |p(x)|$

Hence

$$\sum_{i=0}^{\infty}(\lambda_i - \mu_k)^2 u_{k\,i}^2 < C \qquad (C = \text{const.})^1 \tag{2.3}$$

Suppose that $p(x)$ is a continous function such that the following conditions hold:

1. For eigenvalues and eigenfunctions of operator L holds the asymptotic formulas

$$\lambda_k = \mu_k + O\Big(\frac{1}{k+1}\Big) \qquad \varphi_k = \psi_k + O\Big(\frac{1}{(k+1)^2}\Big)$$

[1] In this paper constants C may be different

2.

$$\int_0^{\pi} p(x)dx = 0$$

.

We shall use condition 1. below estimating.
From inequality (2.3) it follows that

$$\sum_{i=N+1}^{\infty} (\lambda_i - \mu_k)^2 u_{i\,k}^2 < C$$

for all integer N and

$$\sum_{i=N+1}^{\infty} (\lambda_i - \mu_k) u_{i\,k}^2 < C$$

.

Then, it is obvious that

$$\sum_{i=N+1}^{\infty} (\lambda_i - \mu_k)(\lambda_i - \lambda_k) u_{i\,k}^2 < C$$

$$\sum_{i=N+1}^{\infty} (\lambda_i - \lambda_k)^2 u_{i\,k}^2 < C$$

.

From here we obtain

$$\sum_{i=N+1}^{\infty} (\lambda_{N+1} - \mu_k)(\lambda_i - \lambda_k) u_{i\,k}^2 \le \sum_{i=N+1}^{\infty} (\lambda_i - \mu_k)(\lambda_i - \lambda_k) u_{i\,k}^2 < C$$

$$\sum_{i=N+1}^{\infty} (\lambda_i - \lambda_k) u_{i\,k}^2 < \frac{C}{\lambda_{N+1} - \mu_k} \qquad (k \le N) \tag{2.4}$$

Now let us prove formula (2.1). We have

$$(\psi_k,\ L\psi_k) = \left(\sum_{i=0}^{\infty} u_{i\,k}\varphi_i,\ \sum_{i=0}^{\infty} \lambda_i u_{i\,k}\varphi_i \right) = \sum_{i=0}^{\infty} \lambda_i u_{i\,k}^2$$

Summation on k from 0 to N we have

$$(\psi_k,\ L\psi_k) = \sum_{k=0}^{N} \sum_{i=0}^{\infty} \lambda_i u_{i\,k}^2$$

Taking into account $\sum\limits_{i=0}^{\infty} u_{i\,k}^2 = 1$ we have

$$(\psi_k,\ L\psi_k) = \sum_{k=0}^{N} \lambda_k = \sum_{k=0}^{N}\sum_{i=0}^{\infty} \lambda_i u_{k\,i}^2$$

Hence we must prove that

$$\lim_{N\to\infty}\left[\sum_{k=0}^{N}\sum_{i=0}^{\infty}\lambda_i u_{i\,k}^2 - \sum_{k=0}^{N}\sum_{i=0}^{\infty}\lambda_i u_{k\,i}^2\right] = 0 \tag{2.5}$$

we have

$$\sum_{k=0}^{N}\sum_{i=0}^{\infty}\lambda_i u_{i\,k}^2 - \sum_{k=0}^{N}\sum_{i=0}^{\infty}\lambda_i u_{k\,i}^2 = \sum_{k=0}^{N}\sum_{i=N+1}^{\infty}(\lambda_i - \lambda_k)u_{i\,k}^2 + \sum_{k=0}^{N}\sum_{i=N+1}^{\infty}\lambda_k(u_{i\,k}^2 - u_{k\,i}^2) \tag{2.6}$$

Let us calculate first sum on the right side of equality (2.6). For convenience while let $N+1$ be even number then we have

$$\sum_{k=0}^{N}\sum_{i=N+1}^{\infty}(\lambda_i - \lambda_k)u_{i\,k}^2 = \sum_{k=0}^{N-1}\sum_{i=N+1}^{\infty}(\lambda_i - \lambda_k)u_{i\,k}^2 + (\lambda_{N+1} - \lambda_N)u_{N+1\,N}^2 + \sum_{i=N+2}^{\infty}(\lambda_i - \lambda_N)u_{i\,N}^2 \tag{2.7}$$

By inequality (2.4) we shall calculate first and third sum on the right side of equality (2.7), when $N\to\infty$

$$\begin{aligned}\sum_{k=0}^{N-1}\sum_{i=N+1}^{\infty}(\lambda_i - \lambda_k)u_{i\,k}^2 &< \sum_{k=0}^{N-1}\frac{C}{(N+1)^4 - (k+1)^4} = \\ &= \sum_{k=1}^{N}\frac{C}{(N+1)^4 - k^4} \le \\ &\le \frac{1}{(N+1)^4 - N^4} + \int_1^N \frac{dx}{(N+1)^4 - x^4} < \\ &< \frac{1}{N^3} + \frac{N+1}{(N+1)^4}\int_1^N \frac{d(\frac{x}{N+1})}{1-(\frac{x}{N+1})^4} = \end{aligned}$$

$$= \frac{1}{N^3} + \frac{1}{(N+1)^3} \int\limits_{\frac{1}{N+1}}^{\frac{N}{N+1}} \frac{du}{1-u^4} =$$

$$= \frac{1}{N^3} + \frac{1}{(N+1)^3} \int\limits_{\frac{1}{N+1}}^{\frac{N}{N+1}} \frac{1}{2}\left(\frac{1}{1+u^2} + \frac{1}{1-u^2}\right) du \leq$$

$$\leq \frac{1}{N^3} + \frac{1}{2}\frac{1}{(N+1)^3}\left(\frac{N}{N+1} - \frac{1}{N+1}\right) +$$

$$+ \frac{1}{2(N+1)^3} \int\limits_{\frac{1}{N+1}}^{\frac{N}{N+1}} \frac{du}{1-u^2} \sim \frac{\ln N}{N^3} \to 0 \text{ as } N \to \infty \qquad (2.8)$$

and

$$\sum_{i=N+2}^{\infty} (\lambda_i - \lambda_N) u_{iN}^2 < \frac{C}{(N+3)^4 - (N+1)^4} < \frac{C}{(N+2)^3} \to 0 \quad \text{as } N \to \infty \qquad (2.9)$$

Now let us calculate the second term on the right side of equality (2.7) when $N \to \infty$. Assume that $N+1$ is even. Using the condition 1. above, we have

$$(\lambda_{N+1} - \lambda_N) u_{N+1N}^2 \leq \lambda_{N+1} - \lambda_N =$$

$$= (N+1)^4 - (N+1)^4 + O\left(\frac{1}{N+1}\right) \to 0 \quad \text{as } N \to \infty \qquad (2.10)$$

Thus for even number $N+1$ from the expressions (2.7), (2.8), (2.9) and (2.10) we have

$$\lim_{N\to\infty} \sum_{k=0}^{N} \sum_{i=N+1}^{\infty} (\lambda_i - \lambda_k) u_{ik}^2 = 0 \qquad (2.11)$$

Formula (2.11) can be proved for odd number $N+1$ similarly.

Now we shall calculate second sum on the right side of equality (2.6) we have

$$u_{i\,k} + u_{k\,i} = (\varphi_i,\ \psi_k) + (\varphi_k,\ \psi_i) = -(\varphi_i - \psi_i,\ \varphi_k - \psi_k) \qquad (2.12)$$

By equality (2.12) and condition 1., we have

$$|u_{i\,k} + u_{k\,i}| \le \|\varphi_i - \psi_i\| \|\varphi_k - \psi_k\| \le \frac{C}{(i+1)^2(k+1)^2} < \frac{C}{(i+1)(k+1)} \quad (2.13)$$

By using Cauchy-Schwarz inequality we have

$$\sum_{i=N+1}^{\infty} (\lambda_i - \mu_k)|u_{i\,k}^2 - u_{k\,i}^2| = \sum_{i=N+1}^{\infty} (\lambda_i - \mu_k)|u_{i\,k} - u_{k\,i}||u_{i\,k} + u_{k\,i}|$$

$$\le \sqrt{\sum_{i=N+1}^{\infty} |u_{i\,k} + u_{k\,i}|^2} \cdot \sqrt{\sum_{i=N+1}^{\infty} (\lambda_i - \mu_k)^2 |u_{i\,k} - u_{k\,i}|^2}$$

$$< \frac{C}{\sqrt{N+2}(k+1)} \quad (2.14)$$

Hence

$$\sum_{i=N+1}^{\infty} |u_{i\,k}^2 + u_{k\,i}^2| < \frac{C}{(k+1)\sqrt{N+2}(\lambda_N - \mu_k)} \quad (2.15)$$

Now we shall estimate the second sum on the right side of equality (2.6), means that

$$\sum_{k=0}^{N} \lambda_k \sum_{i=N+1}^{\infty} |u_{i\,k}^2 - u_{k\,i}^2| = \lambda_N \sum_{i=N+1}^{\infty} |u_{iN}^2 - u_{Ni}^2| + \sum_{k=0}^{N-1} \lambda_k \sum_{i=N+1}^{\infty} |u_{i\,k}^2 - u_{k\,i}^2|$$

$$= \lambda_N |u_{N+1N}^2 - u_{NN+1}^2| + \lambda_N \sum_{i=N+2}^{\infty} |u_{iN}^2 - u_{Ni}^2| + \sum_{k=0}^{N-1} \lambda_k \sum_{i=N+1}^{\infty} |u_{i\,k}^2 - u_{k\,i}^2| \quad (2.16)$$

By inequality (2.13) we have

$\lambda_N |u_{N+1N}^2 - u_{NN+1}^2| = \lambda_N |u_{N+1N} - u_{NN+1}||u_{N+1N} + u_{NN+1}| \le$

$$\le C(N+1)^4 \frac{1}{(N+2)^2(N+1)^2} |u_{N+1N} - u_{NN+1}| \to 0 \quad \text{as} \quad N \to \infty \quad (2.17)$$

By the expression (2.15) we estimate the second and third sum on the right side of equality (2.16)

$\lambda_N | \sum_{i=N+2}^{\infty} u_{iN}^2 - u_{Ni}^2| <$

$$< C \frac{(N+1)^2}{(N+1)(N+2)[(N+3)^4 - (N+1)^4]} \to 0 \quad \text{as } N \to \infty \quad (2.18)$$

and

$$\sum_{k=0}^{N-1}\lambda_k\sum_{i=N+1}^{\infty}|u_{i\,k}^2-u_{k\,i}^2|<C\sum_{k=0}^{N-1}\frac{(k+1)^2}{(k+1)\sqrt{N+2}[(N+1)^4-(k+1)^4]}$$

$$=C\sum_{k=1}^{N}\frac{k^4}{k\sqrt{N+2}[(N+1)^4-k^4]}<CN^{3-\frac{1}{2}}\sum_{k=1}^{N}\frac{1}{(N+1)^4-k^4}$$

$$\sim CN^{\frac{5}{2}}\frac{\ln N}{(N+1)^3}\to 0 \qquad \text{as} \quad N\to\infty \tag{2.19}$$

From the expressions (2.16), (2.17),(2.18) and (2.19) we have

$$\lim_{N\to\infty}\sum_{k=0}^{N}\sum_{i=N+1}^{\infty}\lambda_k(u_{i\,k}^2-u_{k\,i}^2)=0 \tag{2.20}$$

Thus from the expressions (2.6), (2.11), and (2.20) we have formula (2.5). Hence formula (2.1) have proved.

3 CALCULATION OF THE REGULARIZED TRACE

It is easy to see that

$$(\varphi_n,\ L\varphi_n)=\lambda_n \qquad \text{and} \qquad (\psi_n,\ L\psi_n)=\mu_n+(\psi_n,\ p\psi_n)$$

Putting these into formula (2.1) we have

$$\sum_{n=0}^{\infty}[(\psi_n,\ L\psi_n)-(\varphi_n,\ L\varphi_n)]=\sum_{n=0}^{N}(\mu_n-\lambda_n)+\sum_{n=0}^{N}(\psi_n,\ p\psi_n)\to 0 \quad \text{as } N\to\infty \tag{3.1}$$

Now we shall calculate $\lim\limits_{N\to\infty}\sum\limits_{n=0}^{N}(\psi_n,\ p\psi_n)$

By condition 2. we have for even number N

$$\sum_{n=0}^{N}(\psi_n,p\psi_n)=\frac{1}{\pi}\int_0^{\pi}p(x)dx+\sum_{n=1}^{\frac{N}{2}}\Big(\frac{2}{\pi}\int_0^{\pi}p(x)\sin^2 2nx\,dx+\frac{2}{\pi}\int_0^{\pi}p(x)\cos^2 2nx\,dx\Big)$$

$$= \frac{1}{\pi}\int_0^\pi p(x)dx + \frac{N}{\pi}\int_0^\pi p(x)dx = 0 \tag{3.2}$$

Analogically we have for odd number N

$$\sum_{n=0}^{N}(\psi_n, p\psi_n) = \frac{1}{\pi}\int_0^\pi p(x)dx + \sum_{n=1}^{\frac{N-1}{2}}\Big(\frac{2}{\pi}\int_0^\pi p(x)\sin^2 2nxdx + \frac{2}{\pi}\int_0^\pi p(x)\cos^2 2nxdx\Big) +$$

$$+\frac{2}{\pi}\int_0^\pi p(x)\sin^2 2(\frac{N-1}{2}+1)x\,dx$$

$$= -\frac{1}{\pi}\int_0^\pi p(x)\cos 4(\frac{N+1}{2})x \ \to 0 \quad \text{as } N\to\infty \tag{3.3}$$

From the expressions (3.2) and (3.3) we have

$$\lim_{N\to\infty}\sum_{n=0}^{N}(\psi_n,\ p\psi_n) = 0 \tag{3.4}$$

Hence from the expressions (3.1) and (3.4) we have

$$\lim_{N\to\infty}\sum_{n=0}^{N}(\lambda_n - \mu_n) = 0 \tag{3.5}$$

Thus we have proved the following theorem.

Theorem 3.1 *If $p(x)$ is continous function such that conditions 1. and 2. above are fulfilled, then the formula (3.5) is true.*

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