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Skew Product Action

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Abstract

Suppose that a countable group Γ acts on a nonatomic standard Borel space ergodically and a group G is a locally compact second countable group with Haar measure. For a cocycle α, we prove that if the cocycle α has a dense range in G, then the skew product action of Γ on $X \times_\alpha G$ is also ergodic.

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1 Introduction

Let (X, Ω, μ) be a nonatomic standard Borel space with probability measure μ and G be a locally compact second countable group with Haar measure λ and with Borel field $\mathcal{B}$. Let Γ be a countable automorphism group preserving the measure μ and acting ergodically in X by a Borel action.

Note that a Borel map $\alpha : \Gamma \times X \to G$ is called a *cocycle* if

$$\alpha(\gamma_1\gamma_2, x) = \alpha(\gamma_1, \gamma_2 \cdot x)\alpha(\gamma_2, x)$$

for all $\gamma_1, \gamma_2 \in \Gamma$ and almost everywhere $x \in X$. Another cocycle $\beta : \Gamma \times X \to G$ is said to be *cohomologous* to α if there is a Borel map $\phi : X \to G$ such that for all $\gamma \in G$ and for almost everywhere $x \in X$,

$$\phi(\gamma \cdot x)\alpha(\gamma, x) = \beta(\gamma, x)\phi(x).$$

We may write $\alpha \sim \beta$ and say that α is cohomologous to β. A cocycle is a *cobounary* if it is cohomologous to the trivial one.

Let $\overline{G}$ be a one point compactification of G. An element $g \in G$ ia called an *essential value* of α if for every neighborhood U of g in $\overline{G}$ and for every

$A \in \Omega$ with $\mu(A) > 0$, $\gamma \in \Gamma$ and $B \subset A$ such that $\mu(B) > 0$, $B \cup \gamma B \subset A$ and $\alpha(\gamma, x) \in O$ for all $x \in B$.

If G is noncompact, we put $\overline{G} = G \cup \{\infty\}$. For compact G, let $\overline{G} = G$. The set of all essential values of α is denoted by $\overline{E(\alpha)}$ and we put $E(\alpha) = \overline{E(\alpha)} \cap G$. Then $E(\alpha)$ is a closed subgroup of G. We can say that it means $E(\alpha) = G$ that the cocycle α has a *dense range* in G. That is, for every $A \in \Omega$ with $\mu(A) > 0$ and open subset $O \subset G$, there exist $\gamma \in \Gamma$ and $B \subset A$ such that $\mu(B) > 0$, $B \cup \gamma B \subset A$ and $\alpha(\gamma, x) \in O$ for all $x \in B$.

Theorem 1.1 [8] *Let Γ be a countable measure preserving the measure μ and acting ergodically on (X, Ω, μ). Let G be a locally compact second countable Abelian group. Then the following conditions hold.*

(i) *$\overline{E(\alpha)}$ is a closed nonempty subset of $\overline{G}$.*

(ii) *$E(\alpha) = \overline{E(\alpha)} \cap A$ is a closed subgroup of A.*

(iii) *If $\alpha_1 \sim \alpha_2$, then we have $\overline{E(\alpha_1)} = \overline{E(\alpha_2)}$.*

(iv) *α is a coboundary if and only if $\overline{E(\alpha)} = \{0\}$.*

If G is not Abelian, (iii) and (iv) in Theorem 1 is not true. That is, there are two cocycles α and β such that α and β are cohomologous but $\overline{E(\alpha)}$ is not conjugate to $\overline{E(\beta)}$, where α and β have values in a non Abelian group G (see [1]). However, If cocycles have dense ranges in a subgroup of G, Theorem1 is true.

Theorem 1.2 [2] *Suppose that F and H are closed subgroups of G and that two cocycles $\alpha, \beta : X \times \Gamma \to G$ take values and have dense range in F and H, respectively. If α and β are cohomologous, then F and H are conjugate in G.*

If $E \subset \mathbf{R}^n$ is a measurable set and $x \in E$, then x is called *a point of density* for E if

$$\lim_{r \to 0} \frac{\mu(B(x,r) \cap E)}{\mu(B(x,r))} = 1,$$

where μ is Lebesgue measure and $B(x,r)$ is the ball of radius r about x. Then Lebesgue's Theorem asserts that for any measurable set $E \subset \mathbf{R}^n$, almost every point of E is a point of density for E.

The general type of Lebesgue's Theorem can be founded in [3]. In [3], instead of $\mathbf{R}^n$, X is assumed to be a locally compact separable metrizable space and μ is a $\sigma-$finite measure on X where is positive on open sets and finite on compact sets. The following theorem which may be called the density Theorem for topological groups, is a type in a locally compact group G with a Haar measure λ.

Theorem 1.3 [7] *Suppose that G is a locally compact group with a Haar measure λ and $E \subset G$ is any bounded Borel set. For every $g \in G$ and every bounded neighborhood U of the identity e of G, let*

$$f_U(x) = \frac{\lambda(Ug \cap E)}{\lambda(Ug)},$$

then f_U converges in measure to the characteristic function χ_E on E as U goes to e. In other words, for every positive number $\epsilon > 0$, there exists a bounded neighborhood V of e such that if $U \subset V$, then $\int |f_U - \chi_E| d\mu < \epsilon$.

Furthermore, for every positive number $\epsilon > 0$, there exists a bounded neighborhood U of e such that for almost every point g in E,

$$\frac{\lambda(Ug \cap E)}{\lambda(Ug)} > 1 - \epsilon.$$

2 Ergodicity

Let G be a locally compact second countable group and Γ be a countable group acting on (X, Ω, μ) ergodically. Any cocycle $\alpha : X \times \Gamma \to G$ gives a new action of Γ on the product space $(X \times G, \mu \times \lambda)$ as follows : For any $\gamma \in \Gamma$ and any $(x, g) \in X \times G$,

$$\gamma \cdot (x, g) = (\gamma \cdot x, \alpha(x, \gamma)g).$$

This action of Γ on $(X \times G, \mu \times \lambda)$ is called a skew product or a skew product action. It is common to write a skew product action in the form $(X \times_\alpha G, \Gamma)$. It interests to ask whether the skew product action is ergodic and to find conditions under which the skew product action is ergodic. In [7], K. Schmidt proved that if the cocycle has a dense range in G, the skew product action is ergodic for a countable group Γ and a Abelian group G. He proved also that the converse is true in [8]—. And in the case that Γ is countable and the Γ action on (X, μ) is approximately finite, if the cocycle has a dense range in G, the ergodicity of a skew product actoin has studied. (see [6],[10] and [11].) In this case, we have to note that the group G may not be Abelian.

In this paper, we prove the ergodicity of a skew product action when Γ is countable and Γ-action on (X, μ) is ergodic with the assumption that G is only locally compact group. In Lemma 4, we prove a estimate in G, which is necessary in proving our mail theorem, Theorem 5.

Theorem 2.1 *Suppose that G is a locally compact second countable group and U is an open set in G. For any $\epsilon > 0$, there is a neighborhood W of e in G such that*

$$\lambda\{\, g \in U \mid Wg \subset U\} \geq (1 - \epsilon)\lambda(U).$$

Proof Let $\epsilon > 0$. Since λ is a haar measure, λ is a Radon measure. There is a compact set $K \subset U$ such that $\lambda(U-K) \le \epsilon\lambda(U)$. Since $\lambda(U) = \lambda(U - K) + \lambda(K) \le \epsilon\lambda(U) + \lambda(K)$, we have $\lambda(K) \ge (1-\epsilon)\lambda(U)$.

For all $x \in U$, there is $\delta_x > 0$ such that $B(x, \delta_x) \subset U$. Let $\mathcal{B} = \{\, B(x, \frac{\delta_x}{2}) \,|\, x \in K \,\}$. Then $\mathcal{B}$ is an open covering of K. Since K is compact, there are $x_1, x_2, \cdots, x_n \in K$ such that $K \subset \cup_{i=1}^n B(x_i, \frac{\delta_{x_i}}{2}) \subset U$. Let $\delta = \min\{\, \frac{\delta_{x_i}}{2} \,|\, i = 1, \cdots, n \,\} > 0$.

We will prove that $K \subset \{\, g \in U \mid Wg \subset U\}$, that is, for any $x \in K$, $B(x, \delta) \subset U$. Let $x \in K$ and $y \in B(x, \delta)$. Since $x \in K$, there is $i \in \{1, \cdots, n\}$ such that

$$d(y, x_i) \le d(y, x) + d(x, x_i) < \delta + \frac{\delta_{x_i}}{2} \le \delta_{x_i}.$$

Hence $y \in B(x_i, \delta_{x_i}) \subset U$ and $B(x, \delta) \subset U$.

Let $W = B(0, \delta)$. Since G is a group, we have that for $x \in G$, $Wx = B(x, \delta)$,

$$\lambda\{\, x \in U \,|\, Wx \subset U \,\} \ge \lambda(K) \ge (1-\epsilon)\lambda(U).$$

QED

Theorem 5 is our main theorem. We prove that the skew product action $X \times_\alpha G$ is ergodic with a general assumption of G when $\Gamma-$action on X is ergodic.

Theorem 2.2 *Suppose G is a locally compact second countable group and Γ is a countable group acting on (X, Ω, μ) ergodically. Let $\alpha : X \times \Gamma \to G$ be a cocycle which has a dense range in G. Then the skew product action $X \times_\alpha G$ is ergodic.*

Proof Let A and B be two subsets of $X \times G$ with $(\mu \times \lambda)(A) > 0$ and $(\mu \times \lambda)(B) > 0$. In order to show that the skew product action of Γ on $X \times G$ is ergodic, it will find an element γ of Γ such that $(\mu \times \lambda)(\gamma \cdot (A) \cap B) > 0$.

For $E \subset X \times G$ and $x \in X$, let $E_x = \{g \in G \,|\, (x, g) \in E\}$. It is sufficient to show that there are subsets C_0 and D of X with positive measures and open subsets U and V in G such that for all $y \in \gamma \cdot C_0$,

$$\lambda([\gamma \cdot (A_0) \cap (D \times V)]_y) > \frac{1}{2}\lambda(V) \tag{1}$$

$$\lambda([B \cap (D \times V)]_y) > \frac{1}{2}\lambda(V), \tag{2}$$

where $A_0 = (C_0 \times U) \cap A$. Then (1) and (2) implies that for any $y \in \gamma \cdot C_0$,

$$\begin{aligned} \lambda([\gamma \cdot (A) \cap B]_y) &\ge \lambda([\gamma \cdot (A_0) \cap B]_y) \\ &\ge \lambda([\gamma \cdot (A_0) \cap B \cap (D \times V)]_y) \\ &\ge \lambda([\gamma \cdot (A_0) \cap (D \times V)]_y) + \lambda([B \cap (D \times V)]_y) - \lambda(V) \\ &> \frac{1}{2}\lambda(V) + \frac{1}{2}\lambda(V) - \lambda(V) = 0. \end{aligned}$$

Therefore we have that for any $y \in \gamma \cdot C_0$, $\lambda([\gamma \cdot (A) \cap B]_y) > 0$. By Fubini's Theorem, we have that

$$(\mu \times \lambda)(\gamma \cdot (A) \cap B) \geq \int_{\gamma \cdot C_0} \lambda([\gamma \cdot A \cap B]_y) dy > 0,$$

which completes our theorem.

From now on, we will prove (1) and (2). Let $x_0 \in A$. By the density Theorem for topological groups, there is an open set U in G such that $\lambda(A_{x_0} \cap U) \geq 0.95\, \lambda(U)$. Let $y_0 \in B$. Then we have $\cup_{g \in G}\, g(U \cap A_{x_0}) = G$. That is, $\{g(U \cap A_{x_0}) \,|\, g \in G\}$ is an open covering of G. There is $g_0 \in G$ such that $\lambda(g_0(U \cap A_{x_0}) \cap B_{y_0}) > 0$.

We can choose $g \in g_0(U \cap A_{x_0}) \cap B_{y_0}$ such that is a density point in both of $g_0(U \cap A_{x_0})$ and B_{y_0}. Then there is an open neighborhood V of g in G such that

$$\lambda(g_0(U \cap A_{x_0}) \cap V) \geq 0.95\, \lambda(V) \tag{3}$$
$$\lambda(B_{y_0} \cap V) \geq 0.95\, \lambda(V). \tag{4}$$

We can consider Borel maps

$$\Psi(x) = \frac{\lambda(g_0(A_x \cap U) \cap V)}{\lambda(V)} \quad \textit{and} \quad \Theta(y) = \frac{\lambda(B_y \cap V)}{\lambda(V)}$$

for any x, $y \in X$. We can have compact subsets C and D of X with positive measure such that $x_0 \in C$, $y_0 \in D$, and Ψ and Θ are continuous on C and D, respectively. Moreover we can assume that

$$\lambda(g_0(A_x \cap U) \cap V) \geq 0.9\, \lambda(V) \quad \textit{and} \quad \lambda(B_y \cap V) \geq 0.9\, \lambda(V) \tag{5}$$

for all $x \in C$ and all $y \in D$. The last inequality of (5) implies (2) because $[B \cap (D \times V)]_y = B_y \times V$. In Lemma 5 letting $\epsilon = 0.05$, there is a neighborhood W of e in G such that

$$\lambda\{\, g \in U \mid Wg \subset U\} \geq 0.95\, \lambda(U). \tag{6}$$

Since α has a dense range, there are $C_0 \subset C$ and $\gamma \in \Gamma$ such that

$$\mu(C_0) > 0\,, \;\; \gamma(C_0) \subset D \;\; \textit{and} \;\; \alpha(x, \gamma) \in W g_0 \;\; \textit{for all} \;\; x \in C_0. \tag{7}$$

Let $x \in C_0$ and $\gamma \cdot x = y$. We will show that

$$\begin{aligned} &[\gamma \quad \cdot (A_0) \cap D \times V]_{\gamma \cdot x = y} \\ \supseteq \quad &\alpha(x, \gamma)[(A_0)_x \cap g_0^{-1}\{g \in V \,|\, W g \subset V\,\} \cap g_0^{-1} V]. \end{aligned} \tag{8}$$

Let $F = [(A_0)_x \cap g_0^{-1}\{g \in V \,|\, Wg \subset V\} \cap g_0^{-1}V]$ and $g \in \alpha(x,\gamma)F$. Then there is $h \in F$ such that $g = \alpha(x,\gamma)h$. Since $h \in F$, we have

$$h \in (A_0)_x\,, \quad Wg_0h \subset V \;\text{ and }\; g_0h \in V. \tag{9}$$

Since $g = \alpha(x,\gamma)h$ and $h \in (A_0)_x$, we have that

$$g \in \{(\gamma \cdot x, \alpha(x,\gamma)k) \,|\, (x,k) \in A_0\}_{\gamma\cdot x} = (\gamma \cdot A_0)_{\gamma\cdot x}.$$

Let $g = \alpha(x,\gamma)h = \alpha(x,\gamma)g_0^{-1}g_0h$. Since the cocycle α has a dense subset in G and $x \in C_0$, $\alpha(x,\gamma)g_0^{-1} \in W$ is followed by (7). By (9), we have that $g = \alpha(x,\gamma)g_0^{-1}g_0h \in Wg_0h \subset V$. Therefore it follows that for any $x \in C_0$,

$$g \in (A_0)_{\gamma\cdot x} \cap V = [(A_0) \cap (D \times V)]_{\gamma\cdot x},$$

which prove (8).

Since λ is a haar measure, we have that for any $x \in C_0$,

$$\begin{aligned} & \lambda\left[\alpha(x,\gamma)[(A_0)_x \cap g_0^{-1}\{g \in V \,|\, Wg \subset V\} \cap g_0^{-1}V]\right] \\ = \;& \lambda\left[g_0(A_x \cap U) \cap \{g \in V \,|\, Wg \subset V\} \cap V\right]. \end{aligned}$$

We have that for any $x \in C_0$,

$$\begin{aligned} & \lambda\left[(\gamma \cdot (A_0) \cap D \times V)_{\gamma\cdot x}\right] \\ \geq \;& \lambda\left[\alpha(x,\gamma)[(A_0)_x \cap g_0^{-1}\{g \in V \,|\, Wg \subset V\} \cap g_0^{-1}V]\right] \\ = \;& \lambda\left[g_0(A_x \cap U) \cap \{g \in V \,|\, Wg \subset V\} \cap V\right] \\ \geq \;& \lambda[g_0(U \cap A_x) \cap V] + \lambda\{g \in V \,|\, Wg \subset V\} - \lambda(V) \\ \geq \;& 0.85\,\lambda(V), \end{aligned}$$

where the fourth inequality is (5) and (6). That is, we have that for any $x \in C_0$,

$$\lambda\left[(\gamma \cdot (A_0) \cap D \times V)_{\gamma\cdot x}\right] \geq 0.85\,\lambda(V),$$

which proves (1). QED

References

[1] Arnold L., Cong N. D., Oseledets V., The essential range of a nonabelian cocycle is not a cohomology invariant, *Israel J. Mathematics,* **116** (2000), 71-76.

[2] Danilenko A.I., Quasinormal subrelations of ergodic equivalence relations, *Proc. Amer. Math. Soc.,* **126-11** (1998).

[3] Federer H., *Geometric measure theory,* Springer Verlag, Berlin, 1969.

[4] Golodets V.Ya, Sinel'shchikov S. D.: Existence and uniqueness of cocycles of an ergodic automorphism with dense ranges in amenable groups , *Inst. Low Temp. Phy. and Engin.,* (1983), 19-83.

[5] Golodets V.Ya, Sinel'shchikov S. D., Classification and Structure of Cocycles of amenable ergodic equivalence relations, *J. Funct. Anal.,* **121** (1994), 455-485.

[6] Golodets V.Ya, Cocycles of dynamical systems with dense range, *Dokl. Akad. Nauk Ukrain. SSR Ser.A* **8** (1982), 3-5.

[7] Halmos P. *Measure theory,* Springer Verlag, New York, 1965.

[8] Schmidt K. *Lecture on cocycles of ergodic transformation group,* MacMillan Company of India, New Deli, 1977.

[9] Schmidt K. *Algebraic ideas in ergodic theory-CBMS Regional Conf. Ser. in Math. (76),* Amer. Math. Soc., 1990.

[10] Zimmer R., Amenable ergodic group actions and an applicaiotn to Poisson boundaries of random walks, *J. Funct. Anal.,* **27** (1978), 350 - 372.

[11] Zimmer R., Random walks on compact groups and the existence of cocycles, *Israel J. Math.,* **26** (1977), 84 - 90.

[12] Walters P., *An introduction to ergodic theory,* Wiley, New York, 1949

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