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Topological center of algebra $(\mathcal{A}^*\mathcal{A})^*$

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Abstract

Let $\mathcal{A}$ be a Banach algebra, we prove that the topological center of algebra $(\mathcal{A}^*\mathcal{A})^*$ is identified (up to algebra isomorphism) with the algebra $RM(\mathcal{A})$ of right multipliers of $\mathcal{A}$; i.e., $Z((\mathcal{A}^*\mathcal{A})^*) \cong RM(\mathcal{A})$ with some conditions on $\mathcal{A}$.

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Let $\mathcal{A}$ and be Banach algebra. We can define right and left actions of $\mathcal{A}$ on the dual space $\mathcal{A}^*$ of $\mathcal{A}$ via

$$\langle fa, b\rangle = \langle f, ab\rangle, \qquad \langle af, b\rangle = \langle f, ba\rangle \qquad (a, b \in \mathcal{A}, f \in \mathcal{A}^*).$$

Then $\mathcal{A}^*$ can be made into a Banach $\mathcal{A}$-bimodule.

The second dual space $\mathcal{A}^{**}$ of a Banach algebra $\mathcal{A}$ admits a Banach algebra products known as first (left) Arens product. We briefly recall the definition of this product.

For $m, n \in \mathcal{A}^{**}$, their first (left) Arens product indicated by $m \,\Box\, n$ is given by

$$\langle m \,\Box\, n, f\rangle = \langle m, nf\rangle \quad (f \in \mathcal{A}^*),$$

where $nf \in \mathcal{A}^*$ is defined by

$$\langle nf, a\rangle = \langle n, fa\rangle \quad (a \in \mathcal{A}).$$

For each $n \in \mathcal{A}^{**}$, the mapping $m \mapsto m \,\Box\, n$ is weak*-weak* continuous. However for certain n, the mapping $m \mapsto n \,\Box\, m$ may fail to be weak* continuous. Due to this lack of symmetry the left topological center $Z(\mathcal{A}^{**})$ of $\mathcal{A}^{**}$ is defined by

$$Z(\mathcal{A}^{**}) = \{m \in \mathcal{A}^{**} : n \mapsto m \,\Box\, n \text{ is weak}^*\text{-weak}^* \text{ continuous}\}.$$

(For more information see [1]). Let

$$\mathcal{A}^*\mathcal{A} = \{fa : f \in \mathcal{A}^*, a \in \mathcal{A}\}, \qquad \mathcal{A}\mathcal{A}^* = \{af : a \in \mathcal{A}, f \in \mathcal{A}^*\}.$$

Then $\mathcal{A}^*\mathcal{A}$ is a closed subalgebra of $\mathcal{A}^*$. Also $(\mathcal{A}^*\mathcal{A})^*$ is a Banach algebra as a closed subalgebra of $\mathcal{A}^{**}$ with the first Arens product. The topological center $Z((\mathcal{A}^*\mathcal{A})^*)$ of $(\mathcal{A}^*\mathcal{A})^*$ is defined by

$$Z((\mathcal{A}^*\mathcal{A})^*) := \{m \in (\mathcal{A}^*\mathcal{A})^* : n \mapsto m \,\Box\, n \text{ is weak}^*\text{-weak}^* \text{ continuous}\}.$$

Neufang in [3] has shown that $Z((\mathcal{A}^*\mathcal{A})^*) \cong RM(\mathcal{A})$ holds for some large classes of Banach algebras. In this paper we fined a necessary and sufficient condition for the relation $Z((\mathcal{A}^*\mathcal{A})^*) \cong RM(\mathcal{A})$ when $\mathcal{A}$ is commutative. We show that if the commutative Banach algebra $\mathcal{A}$ is an ideal in $\mathcal{A}^{**}$, then $Z((\mathcal{A}^*\mathcal{A})^*) \cong RM(\mathcal{A})$. Let $HOM_{\mathcal{A}}(\mathcal{A}^*, \mathcal{A}^*)$ be the Banach algebra of right $\mathcal{A}-$ module homomorphisms on $\mathcal{A}^*$. Then we have the follows:

1.Theorem. Let $\mathcal{A}$ be a Banach algebra. Then there exists an anti isomorphism between $RM(\mathcal{A})$ and $Z(HOM_{\mathcal{A}}(\mathcal{A}^*, \mathcal{A}^*))$ the subalgebra of $weak^*-weak^*-$ continuous elements of $HOM_{\mathcal{A}}(\mathcal{A}^*, \mathcal{A}^*)$.

proof. Let $* : BL(\mathcal{A}, \mathcal{A}) \to BL(\mathcal{A}^*, \mathcal{A}^*)$ maps every $T \in RM(\mathcal{A})$ to T^* the adjoint of T. We show that $\Phi := *|_{RM(\mathcal{A})}$ the restriction of $*$ to $RM(\mathcal{A})$ is an anti isomorphism between $RM(\mathcal{A})$ and $Z(HOM_{\mathcal{A}}(\mathcal{A}^*, \mathcal{A}^*))$. Suppose $T \in RM(\mathcal{A})$, then we know that T^* is $weak^* - weak^*-$ continuous. We show that $T^* \in HOM_{\mathcal{A}}(\mathcal{A}^*, \mathcal{A}^*)$. For every $f \in \mathcal{A}^*$ and $a, b \in \mathcal{A}$ we have

$$\begin{aligned} \langle T^*(fa), b\rangle &= \langle fa, Tb\rangle \\ &= \langle f, aTb\rangle \\ &= \langle f, T(ab)\rangle \\ &= \langle T^*f, ab\rangle \\ &= \langle (T^*f)a, b\rangle. \end{aligned}$$

Then $T^* \in HOM_{\mathcal{A}}(\mathcal{A}^*, \mathcal{A}^*)$. For the convers let $S \in HOM_{\mathcal{A}}(\mathcal{A}^*, \mathcal{A}^*)$ be $weak^*-weak^*-$ continuous, then there exists $T \in BL(\mathcal{A}, \mathcal{A})$ such that $T^* = S$. We show that $T \in RM(\mathcal{A})$. Suppose $a, b \in \mathcal{A}$ and $f \in \mathcal{A}^*$ then

$$\begin{aligned} \langle f, T(ab)\rangle &= \langle S(f), ab\rangle \\ &= \langle S(f)a, b\rangle \\ &= \langle S(fa), b\rangle \\ &= \langle fa, Tb\rangle \\ &= \langle f, aTb\rangle. \end{aligned}$$

Therefore by Hahn-Banach theorem, $T(ab) = aTb$ and $T \in RM(\mathcal{A})$. It is easy to see that Φ is a surjective map and anti homomorphism.

Let $\mathcal{A}$ has bai (bounded approximate identity), then $\mathcal{A}^{**}$ has right identity. Let E be a right identity of $\mathcal{A}^{**}$ and let $\Psi : HOM_{\mathcal{A}}(\mathcal{A}^*, \mathcal{A}^*) \to (\mathcal{A}^*\mathcal{A})^*$ defined by

$$\langle \Psi(T), f\rangle = \langle E, T(f)\rangle \qquad (T \in HOM_{\mathcal{A}}(\mathcal{A}^*, \mathcal{A}^*), f \in \mathcal{A}^*\mathcal{A}).$$

Baker , Lau and Pym in [2; Theorem 1.1] have shown that Ψ is an (algebric) isomorphism between $HOM_{\mathcal{A}}(\mathcal{A}^*, \mathcal{A}^*)$ and $(\mathcal{A}^*\mathcal{A})^*$. We would show that the restriction of Ψ is an isomorphism between $Z(HOM_{\mathcal{A}}(\mathcal{A}^*, \mathcal{A}^*))$ and $Z((\mathcal{A}^*\mathcal{A})^*)$ with some conditions on $\mathcal{A}$.

2.Lemma. Let $\mathcal{A}$ be a commutative Banach algebra with a bai, and let $\Psi : HOM_{\mathcal{A}}(\mathcal{A}^*, \mathcal{A}^*) \to (\mathcal{A}^*\mathcal{A})^*$ be the above isomorphism, then

$$\Psi(Z(HOM_{\mathcal{A}}(\mathcal{A}^*, \mathcal{A}^*))) \subseteq Z((\mathcal{A}^*\mathcal{A})^*).$$

proof. Let $T \in Z(HOM_{\mathcal{A}}(\mathcal{A}^*, \mathcal{A}^*))$ then by theorem 1 there exists $S \in RM(\mathcal{A})$ such that $S^* = T$. For $a, b, c \in \mathcal{A}$ and $f \in \mathcal{A}^*$, we have

$$\begin{aligned}
\langle fS(ab), c\rangle &= \langle f, S(ab)c\rangle \\
&= \langle f, S(abc)\rangle \\
&= \langle f, aS(bc)\rangle \\
&= \langle fa, S(bc)\rangle \\
&= \langle T(fa), bc\rangle \\
&= \langle T(fa)b, c\rangle.
\end{aligned}$$

Then we have

$$fS(ab) = T(fa)b \qquad (1).$$

Let now $b'' \in (\mathcal{A}^*\mathcal{A})^*$, then we have

$$\begin{aligned}
\langle T(b''fa), b\rangle &= \langle b''fa, Sb\rangle \\
&= \langle b''f, aSb\rangle \\
&= \langle b''f, S(ab)\rangle \\
&= \langle b'', fS(ab)\rangle \\
&= \langle b'', T(fa)b\rangle \quad (by(1)) \\
&= \langle b''T(fa), b\rangle.
\end{aligned}$$

Therefor

$$T(b''fa) = b''T(fa) \quad (2).$$

We show that $\Psi(T) \in Z((\mathcal{A}^*\mathcal{A})^*)$. Let $b''_\alpha \xrightarrow{weak^*} b''$ in $(\mathcal{A}^*\mathcal{A})^*$. Then for every $f \in \mathcal{A}^*$ and $a \in \mathcal{A}$ we have

$$\begin{aligned}
lim_\alpha \langle \Psi(T)b''_\alpha, fa \rangle &= lim_\alpha \langle \Psi(T), b''_\alpha fa \rangle \\
&= lim_\alpha \langle E, T(b''_\alpha fa) \rangle \\
&= lim_\alpha \langle Eb''_\alpha, T(fa) \rangle \quad (by(2)) \\
&= lim_\alpha \langle b''_\alpha, T(fa) \rangle \\
&= \langle b'', T(fa) \rangle \\
&= \langle Eb'', T(fa) \rangle \\
&= \langle \Psi(T), b''fa \rangle.
\end{aligned}$$

Thus $\Psi(T)b''_\alpha \xrightarrow{weak^*} \Psi(T)b''$ in $(\mathcal{A}^*\mathcal{A})^*$.

3.Proposition. Let $\mathcal{A}$ be commutative Banach algebra without order, with a bai, and let $\Psi : HOM_{\mathcal{A}}(\mathcal{A}^*, \mathcal{A}^*) \to (\mathcal{A}^*\mathcal{A})^*$ be the above isomorphism, then $M(\mathcal{A}) \cong Z((\mathcal{A}^*\mathcal{A})^*)$ if and only if $\Psi^{-1}(Z((\mathcal{A}^*\mathcal{A})^*)) \subseteq Z(HOM_{\mathcal{A}}(\mathcal{A}^*, \mathcal{A}^*))$.

proof. Let $\mathcal{A}$ be commutative and without order, then $RM(\mathcal{A})(= M(\mathcal{A})$ is commutative. Thus Ψ is a homomorphism and the proof is straightforward.

4.Corollary. Let $\mathcal{A}$ be a commutative Banach algebra without order with a bai. If $\mathcal{A}$ is an ideal in $\mathcal{A}^{**}$, then $M(\mathcal{A}) \cong Z((\mathcal{A}^*\mathcal{A})^*)$.

proof. Let $z \in Z((\mathcal{A}^*\mathcal{A})^*)$. then we show that $\Psi^{-1}(z) \in Z(HOM_{\mathcal{A}}(\mathcal{A}^*, \mathcal{A}^*))$. Suppose that $f_\alpha \xrightarrow{weak^*} f$ in $\mathcal{A}^*$, then for every $a \in \mathcal{A}$, by [2; Theorem 1.1],we have

$$\begin{aligned}
lim_\alpha \langle \Psi^{-1}(z)(f_\alpha), a \rangle &= lim_\alpha \langle zf_\alpha, a \rangle \\
&= lim_\alpha \langle z, f_\alpha a \rangle \\
&= lim_\alpha \langle az, f_\alpha \rangle \\
&= \langle az, f \rangle \\
&= \langle \Psi^{-1}(z)(f), a \rangle.
\end{aligned}$$

and the proof is complete.

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