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A remark on the instability of positive solutions to a diffusive logistic equation

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Abstract

We study the stability of positive solution to the boundary value problem

$$\begin{cases} -\Delta_p u = \lambda m(x)(u^{\gamma-1} - u^{p-1} - ch(x)), & x \in \Omega, \\ Bu = 0, & x \in \partial\Omega, \end{cases}$$

where Δ_p denotes the p-Laplacian operator defined by $\Delta_p z = div(|\nabla z|^{p-2}\nabla z)$; $p > 1$, $\gamma(> p)$, Ω is a bounded domain in $R^N(N \geq 1)$ with smooth boundary $Bu(x) = \alpha g(x)u + (1-\alpha)\frac{\partial u}{\partial n}$ where $\alpha \in [0,1]$, $g : \partial\Omega \longrightarrow R^+$ with $g = 1$ when $\alpha = 1$, and c, λ are positive constants, the weight $m(x)$ satisfies $m(x) \in C(\Omega)$, $m(x) > 0$ for all $x \in \Omega$ and $h : \overline{\Omega} \longrightarrow R$ is a $C^{1,\alpha}(\overline{\Omega})$ function satisfying $h(x) > 0$ for $x \in \Omega$, $\max h(x) = 1$ for $x \in \overline{\Omega}$ and $h(x) = 0$ for $x \in \partial\Omega$. We shall establish that every positive solution is lineary unstable.

Keywords: Diffusive logistic equation; harvesting; linearized stability of positive solutions; p-Laplacian.
Mathematics Subject Classification: 35J60, 35B30, 35B40

1 Introduction

In this paper, we consider the stability of positive solution to the boundary value problem

$$-\Delta_p u = \lambda m(x)(u^{\gamma-1} - u^{p-1} - ch(x)), \qquad x \in \Omega, \tag{1}$$

$$Bu = 0, \qquad x \in \partial\Omega, \tag{2}$$

where Δ_p denotes the p-Laplacian operator defined by $\Delta_p z = div(|\nabla z|^{p-2}\nabla z)$; $p > 1$, $\gamma(> p)$, Ω is a bounded domain in $R^N(N \geq 1)$ with smooth boundary

$Bu(x) = \alpha g(x)u + (1-\alpha)\frac{\partial u}{\partial n}$ where $\alpha \in [0,1]$, $g : \partial\Omega \longrightarrow R^+$ with $g = 1$ when $\alpha = 1$, i.e., the boundary condition may be of Dirichlet, Neumann or mixed type, and c, λ are positive constants, the weight $m(x)$ satisfies $m(x) \in C(\Omega)$, $m(x) > 0$ for all $x \in \Omega$ and $h : \overline{\Omega} \longrightarrow R$ is a $C^{1,\alpha}(\overline{\Omega})$ function satisfying $h(x) > 0$ for $x \in \Omega$, $\max h(x) = 1$ for $x \in \overline{\Omega}$ and $h(x) = 0$ for $x \in \partial\Omega$.

Equation (1) arises in the studies of population biology of one species. Here, u is the population density, $\lambda m(x)(u^{\gamma-1} - u^{p-1})$ represent the logistic growth and $\lambda m(x)ch(x)$ representing the rate of harvesting (see [1, 4]). In [2], instability of such solutions was proven when $p = 2$ (the Laplacian operator), $\gamma = 3$ and $c = 0$ (non-harvesting case). The purpose of this paper is to extend this study to the p-Laplacian case with constant yield harvesting. However studying the instability of positive solutions in this case is significantly harder. For existence results of positive solutions for Eq. (1) see [3].

We recall that, if u be any nonnegative solution of

$$\begin{cases} -\Delta_p u = g(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{3}$$

then the linearized equation about u is

$$\begin{cases} -(p-1)\, div\, (|\nabla u|^{p-2}\nabla\phi) - g_u(x,u)\phi = \mu\phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega, \end{cases} \tag{4}$$

where $g_u(x,u)$ denotes the partial derivative of $g(x,u)$ with respect to u. Eq. (4) obtained from the formal derivative of the operator Δ_p.

Definition 1.1. We call a solution u of (3) a linearly stable solution if all eigenvalues of (4) are strictly positive, which can be inferred if the principal eigenvalue $\mu_1 > 0$. Otherwise u is linearly unstable.

2 Main result

In this section, we shall prove the instability of solution u by showing that the principal eigenvalue μ_1, of the equation linearized about u is negative; the instability of u then follows from the well-known principle of linearized stability (see [5]). Our main result is formulate in the following theorem.

Theorem 2.1. Every positive solution of (1)-(2) is linearly unstable.

Proof. From (4) the linearized equation about u is

$$-(p-1)\, div\, (|\nabla u|^{p-2}\nabla\phi) - \lambda m(x)[(\gamma-1)u^{\gamma-2} - (p-1)u^{p-2}]\phi = \mu\phi, \qquad x \in \Omega, \tag{5}$$

$$B\phi = 0, \qquad x \in \partial\Omega. \tag{6}$$

Let μ_1 be the principal eigenvalue and let $\psi(x)$ (≥ 0) be a corresponding eigenfunction. Multiplying (1) by $(p-1)\psi(x)$ and (5) by u, then subtracting and integrating over Ω, we obtain

$$(p-1)\int_\Omega [u\, div(|\nabla u|^{p-2}\nabla\psi) - \psi(x)\, div(|\nabla u|^{p-2}\nabla u)]\, dx$$

$$+\int_\Omega \lambda\, \psi(x)\, m(x)[(\gamma-p)\ u^{\gamma-1} + (p-1)ch(x)]\, dx$$

$$= -\mu_1 \int_\Omega \psi(x)u(x)\, dx. \tag{7}$$

But by green's first identity

$$\int_\Omega u\, div(|\nabla u|^{p-2}\nabla\psi)dx = \int_\Omega u|\nabla u|^{p-2}(\Delta\psi)dx + \int_\Omega u\nabla\psi\nabla(|\nabla u|^{p-2})dx$$

$$= -\int_\Omega \nabla(u|\nabla u|^{p-2})\nabla\psi(x)dx$$

$$+\int_\Omega u\nabla\psi\nabla(|\nabla u|^{p-2})dx + \int_{\partial\Omega} u|\nabla u|^{p-2}(\frac{\partial\psi}{\partial n})ds$$

$$= -\int_\Omega |\nabla u|^{p-2}(\nabla u\nabla\psi)dx + \int_{\partial\Omega} u|\nabla u|^{p-2}(\frac{\partial\psi}{\partial n})ds, \tag{8}$$

and

$$\int_\Omega \psi(x)div(|\nabla u|^{p-2}\nabla u)dx = \int_\Omega \psi(x)|\nabla u|^{p-2}(\Delta u)dx + \int_\Omega \psi(x)\nabla u\nabla(|\nabla u|^{p-2})dx$$

$$= -\int_\Omega \nabla(\psi|\nabla u|^{p-2})\nabla u dx$$

$$+\int_\Omega \psi(x)\nabla u\nabla(|\nabla u|^{p-2})dx + \int_{\partial\Omega} \psi(x)|\nabla u|^{p-2}(\frac{\partial u}{\partial n})ds$$

$$= -\int_\Omega |\nabla u|^{p-2}(\nabla u\nabla\psi)dx + \int_{\partial\Omega} \psi(x)|\nabla u|^{p-2}(\frac{\partial u}{\partial n})ds. \tag{9}$$

By using (8) – (9) in (7) we get

$$-\mu_1 \int_\Omega \psi(x)u(x)dx = \lambda \int_\Omega \psi(x)\, m(x)\, [(\gamma-p)u^{\gamma-1} + (p-1)ch(x)]\, dx$$

$$+\int_{\partial\Omega} |\nabla u|^{p-2}[u(\frac{\partial\psi}{\partial n}) - \psi(s)(\frac{\partial u}{\partial n})]ds. \tag{10}$$

We notice that when $\alpha = 1$(then $h = 1$)we have $Bu = u = 0$ for $s \in \partial\Omega$ and also we have $\psi = 0$ for $s \in \partial\Omega$. Hence,

$$\int_{\partial\Omega} |\nabla u|^{p-2}[u(\frac{\partial\psi}{\partial n}) - \psi(s)(\frac{\partial u}{\partial n})]ds = 0, \tag{11}$$

and when $\alpha \neq 1$, we have

$$\int_{\partial\Omega} |\nabla u|^{p-2}[u(\frac{\partial\psi}{\partial n}) - \psi(s)(\frac{\partial u}{\partial n})]ds = \int_{\partial\Omega} |\nabla u|^{p-2}\{\frac{\alpha g\psi(s)}{(1-\alpha)}\}(u-u)ds = 0. \tag{12}$$

By using $(11) - (12)$ in (10) we get

$$-\mu_1 \int_\Omega \psi(x)\, u(x)\, dx = \lambda \int_\Omega \psi(x)\, m(x)\, [(\gamma - p)u^{\gamma-1} + (p-1)ch(x)]\, dx > 0. \tag{13}$$

But $\psi > 0$ for $x \in \Omega$ and $u > 0$. Hence , it is easy to see that $\mu_1 < 0$ and the result follows (see [5]). $\diamond$

References

[1] G.A. Afrouzi, K.J. Brown. On diffusive logistic equation. J. Math. Anal. Appl., 225, (1998),no. 1, 326-339.

[2] G.A. Afrouzi, S.H. Rasouli. Instability of nonnegative solutions for a boundary value problem with indefinite weight function. Global. J. Pure. Appl. Math. Vol 1, no. 1, (2005), pp. 9-12.

[3] G.A. Afrouzi, S.H. Rasouli. Population models involving the p-Laplacian with indefinite weight and constant yeild harvesting. (to appear in Chaos, Solutions and Fractals)

[4] S. Oruganti, J. Shi, and R. Shivaji. Logistic equation with the p-Laplacian and constant yeild harvesting. Abstr. Appl. Anal., 9 (2004), pp. 723-727.

[5] D.H. Satinger. Monotone methods in nonlinear elliptic and parabolic boundary value problems, Indiana Univ. Math. J., 21 (11)(1972), pp. 979-1000.

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