Int. J. Contemp. Math. Sci., Vol. 1, 2006, no. 5, 213 - 217

Stability properties of positive solutions for a boundary value problem

G. A. Afrouzi and S. H. Rasouli

Department of Mathematics, Faculty of Basic Sciences
Mazandaran University, Babolsar, Iran
afrouzi@umz.ac.ir

Abstract

We consider the boundary value problem

$$\begin{cases} -\Delta u = \lambda m(x) f(u) + g(x,u), & x \in \Omega, \\ Bu = 0, & x \in \partial\Omega, \end{cases}$$

where Δ denotes the Laplacian operator, Ω is a bounded domain in $R^N (N \geq 1)$ with smooth boundary $Bu(x) = \alpha h(x) u + (1-\alpha)\frac{\partial u}{\partial n}$ where $\alpha \in [0,1]$, $h : \partial\Omega \longrightarrow R^+$ with $h = 1$ when $\alpha = 1$, $\lambda > 0$, the weight $m(x)$ satisfies $m(x) \in C(\Omega)$, $m(x) > 0$ for all $x \in \Omega$, $f \in C^2[0,\infty)$ and $g : \Omega \times [0,\infty) \longrightarrow R$ is a continuous function. We provide a simple proof to stablish that every positive solution is lineary unstable under certain conditions.

Keywords: Instability, Positive solutions.
Mathematics Subject Classification: 35J60, 35B30, 35B40

1 Introduction

In this paper, we study the stability of positive solutions to the boundary value problem

$$-\Delta u = \lambda m(x) f(u) + g(x,u), \qquad x \in \Omega, \tag{1}$$

$$Bu = 0, \qquad x \in \partial\Omega, \tag{2}$$

where Δ denotes the Laplacian operator, Ω is a bounded domain in $R^N (N \geq 1)$ with smooth boundary $Bu(x) = \alpha h(x) u + (1-\alpha)\frac{\partial u}{\partial n}$ where $\alpha \in [0,1]$, $h : \partial\Omega \longrightarrow R^+$ with $h = 1$ when $\alpha = 1$, i.e., the boundary condition may be of Dirichlet, Neumann or mixed type, $\lambda > 0$ is a constant, the weight

$m(x)$ satisfies $m(x) \in C(\Omega)$, $m(x) > 0$ for all $x \in \Omega$, $f \in C^2[0, \infty)$, and $g : \Omega \times [0, \infty) \longrightarrow R$ is a continuous function.

Shivaji and his coauthors have altogether proved that if $f'' > 0$ and $f(0) \leq 0$, then every non-trivial nonnegative solution to the elliptic boundary value problem

$$\begin{cases} -\Delta u = \lambda f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{3}$$

is unstable. In fact, they proved that the first eigenvalue of the linearized problem is negative, which implies instability. They first consider the monotone case, i.e., $f' > 0$ in [1]. The non-monotone case was first proved by Tertikas [5] using sub- and supersolution . This prof was simplified by Maya and Shivaji [3], reducing the problem to the monotone one via the decomposition of f to a monotone and a linear function. The purpose of this paper is to extend this result to Eq. $(1)-(2)$, under certain conditions. Karatson and Simon gave a direct proof of the result in [2]. This can be summed up in the theorem below.

Theorem 1.1. Let $f : R \longrightarrow R$ be a twice continuously differentiable function, then

(i) if $f'' > 0$ and $f(0) \leq 0$, then every nontrivial nonnegative solution of (3) is unstable. while
(ii) if $f'' < 0$ and $f(0) \geq 0$, then every nontrivial nonnegative solution of (3) is stable.

We call a function g strictly convex (or concave) if $g'' \geq$ $(g'' \leq)$, respectively, and not constant zero on any subinterval.

We recall that, if u be any nonnegative solution of

$$\begin{cases} -\Delta u = g(x, u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{4}$$

then the linearized equation about u is

$$\begin{cases} -\Delta \phi - g_u(x, u)\phi = \mu\phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega, \end{cases} \tag{5}$$

where $g_u(x, u)$ denotes the partial derivative of $g(x, u)$ with respect to u.

Definition 1.2. We call a solution u of (4) a linearly stable solution if all eigenvalues of (5) are strictly positive, which can be inferred if the principal eigenvalue $\mu_1 > 0$. Otherwise u is linearly unstable.

2 Stability results

In this section, we shall prove the instability of positive solution u by showing that the principal eigenvalue μ_1, of the equation linearized about u is negative, the instability of u then follows from the well-known principle of linearized stability (see [4]).
Our main result is formulate in the following theorem.

Theorem 2.1. If $f'' > 0$, $f(0) \leq 0$ and $u \to g(x,u)$ be strictly convex and $g(x,0) \leq 0$ for all fixed $x \in \Omega$, then every positive solution of $(1)-(2)$ is linearly unstable.

Proof. Let u be any nontrivial nonnegative stationary solution of $(1)-(2)$, then from (5) the linearized equation about u is

$$-\Delta\phi - [\lambda m(x)f'(u) - g_u(x,u)]\phi = \mu\phi, \qquad x \in \Omega, \tag{6}$$

$$B\phi = 0, \qquad x \in \partial\Omega. \tag{7}$$

Let μ_1 be the principal eigenvalue and let $\psi(x)(\geq 0)$ be a corresponding eigenfunction.Multiplying (1) by $\psi(x)$ and (6) by u, then subtracting and integrating over Ω, we obtain

$$\begin{aligned}\int_\Omega [u\Delta\psi - \psi(x)\Delta u)]dx + \lambda\int_\Omega m(x)\psi(x)[uf'(u) - f(u)]dx \\ + \int_\Omega \psi(x)[ug_u(x,u) - g(x,u)]dx \\ = -\mu_1\int_\Omega \psi(x)u(x)dx.\end{aligned} \tag{8}$$

But by green's first identity

$$\int_\Omega u\Delta\psi dx = -\int_\Omega(\nabla u\nabla\psi)dx + \int_{\partial\Omega} u(\frac{\partial\psi}{\partial n})ds, \tag{9}$$

and

$$\int_\Omega \psi(x)\Delta u dx = -\int_\Omega(\nabla u\nabla\psi)dx + \int_{\partial\Omega}\psi(x)(\frac{\partial u}{\partial n})ds. \tag{10}$$

By using $(9)-(10)$ in (8) we get

$$\begin{aligned}-\mu_1\int_\Omega \psi(x)u(x)dx = \lambda\int_\Omega m(x)\psi(x)[uf'(u) - f(u)]dx \\ + \int_\Omega \psi(x)[ug_u(x,u) - g(x,u)]dx\end{aligned}$$

$$+\int_{\partial\Omega}[u(\frac{\partial\psi}{\partial n})-\psi(s)(\frac{\partial u}{\partial n})]ds. \tag{11}$$

We notice that when $\alpha = 1$(then $h = 1$)we have $Bu = u = 0$ for $s \in \partial\Omega$ and also we have $\psi = 0$ for $s \in \partial\Omega$. Hence,

$$\int_{\partial\Omega}[u(\frac{\partial\psi}{\partial n})-\psi(s)(\frac{\partial u}{\partial n})]ds = 0, \tag{12}$$

and when $\alpha \neq 1$, we have

$$\int_{\partial\Omega}[u(\frac{\partial\psi}{\partial n})-\psi(s)(\frac{\partial u}{\partial n})]ds = \int_{\partial\Omega}\{\frac{\alpha h\psi(s)}{(1-\alpha)}\}(u-u)ds = 0. \tag{13}$$

By using $(12)-(13)$ in (11) we get

$$-\mu_1\int_{\Omega}\psi(x)u(x)dx = \lambda\int_{\Omega}m(x)\psi(x)[uf'(u)-f(u)]dx$$

$$+\int_{\Omega}\psi(x)[ug_u(x,u)-g(x,u)]dx \tag{14}$$

Our assumption implies that $uf'(u)-f(u) > 0$ for $u \in R^+$ also, $ug_u(x,u)-g(x,u) > 0$ for $u \in R^+$. Thus, we have

$$-\mu_1\int_{\Omega}\psi(x)u(x)dx > 0. \tag{15}$$

Hence, it is easy to see that $\mu_1 < 0$ and the result follows (see [4]). $\diamond$

Corollary 2.3. Assume that in Theorem 2.1 we have $f'' < 0$, $f(0) \geq 0$ and $u \to g(x,u)$ be strictly concave and $g(x,0) \geq 0$ for all fixed $x \in \Omega$, then every positive solution of $(1)-(2)$ is linearly stable.

Proof. The proof proceeding is identical to the proof of Theorem 2.1. In fact, instead of (15) we get

$$-\mu_1\int_{\Omega}\psi(x)u(x)dx < 0, \tag{16}$$

but $\psi > 0$ for $x \in \Omega$, $u > 0$ and hence $\mu_1 > 0$. This completes the proof. $\diamond$

Remark 1.2. Recently in [6], the author study the stability of nonnegative stationary solutions of symmetric cooperative semilinear systems with some convex (resp, concave) nonlinearity condition.

References

[1] K.J. Brown and R. Shivaji. Instability of nonnegative solution for a class of semipositone problems. Proc. Amer. Math. Soc., 112 (1991), pp. 121-124.

[2] J. Karatson and P.L. Simon. On the stability properties of nonnegative solutions of semilinear problems with convex or concave nonlinearity. J. Comput. Appl. Math., 131 (2001), pp. 497-501.

[3] C. Maya and R. Shivaji. Instability of nonnegative solutions for a class of semilinear elliptic boundary value problems. J. Comput. Appl. Math., 88 (1998), pp. 125-128.

[4] D.H. Satinger. Monotone methods in nonlinear elliptic and parabolic boundary value problems, Indiana Univ. Math. J., 21 (11)(1972), pp. 979-1000.

[5] A. Tertikas. Stability and instability of positive solutions of semi-positone problems, Proc. AMS., 114 (4) (1992), pp. 1035-1040.

[6] I. Voros. Stability properties of nonnegative solutions of semilinear symmetric cooperative systems. Electronic Jour. Diff. Eqns., 105 (2004), pp. 1-6.

Received: October 19, 2005