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Mixed Boundary Value Problems in Semi - infinite Strip

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Abstract. we consider two mixed boundary value problems for a strip. One of the physical problem it represents is the steady flow of heat into the strip through the segment$0 < u < \pi$ of the u - axis and the entire v- axis not being perfectly insulated. In order to solve the problem, we have developed a numerical procedure which is in our opinion a simple technique.

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1-1 : Introduction

Mathematical models of continuum phenomena often lead to Laplace' equation. Laplace' equation occurs in the analysis of equilibrium phenomena which arise in a wide variety of physical contexts including Electrostatics, Fluid Dynamics, Heat conduction and Diffusion. In most mathematical problems of physical interest , the desired (potential) function must satisfy not only Laplace' equation in some specified region, but must also exhibit certain prescribed behavior along the surface that bounds the region. Occasionally, the boundary conditions are uniform .That is to say, the value of the function alone or some linear combination of these is specified a priori on the boundary. In this work, we consider two mixed boundary value problems for a strip. One of the physical problem it represents is the steady flow of heat into the strip through the segment $0 < u < \pi$ of the u - axis and the entire v- axis not being perfectly insulated. In order to solve the problem, we have developed a numerical procedure which is in our opinion a simple technique.

Problem 0.1. *: Find the solution of the following mixed boundary value problem*

P.D.E. $\Delta\Phi = 0, \qquad u > 0\ ,\ v > 0$

B.C.

$$
\begin{aligned}
&1)\quad \Phi_u(0,v) + h\Phi(0,v) = 0, && v > 0\\
&2)\quad \Phi_v(u,0) = f(u), && 0 < u < \pi\\
&3)\quad \Phi(\pi,v) = 0, && v > 0
\end{aligned}
$$

Solution: By separation of variables method and using boundary conditions we get

$$
\begin{aligned}
A(\lambda)\sin\lambda\pi + B(\lambda)\cos\lambda\pi &= 0\\
\lambda A(\lambda)\cos 0\lambda + hB(\lambda)\cos 0\lambda &= 0.
\end{aligned}
\qquad (1-1)
$$

Thus, we get $\lambda A(\lambda) + hB(\lambda) = 0$
or, $\tan\lambda_n\pi = -\frac{B(\lambda_n)}{A(\lambda_n)}$ which implies that

$$\tan\lambda_n\pi = \frac{\lambda_n}{h}, \qquad \lambda_n > 0. \qquad (1-2)$$

In order to have first non - zero eigenvalue in the interval (0 , 1/2)

$$0 < h\pi < 1, \qquad (1-3)$$

and also, we have the following relation for the n-th eigenvalue :

$$\lambda_n < n - \frac{1}{2} \qquad n = 1, 2, 3, \cdots .$$

It is interesting to note that as n increases, the intersecting points more closely approaches the position of the vertical tangent function .
So we have the following approximate (asymptotic) formula for the eigenvalues

$$\lambda_n \cong n - \frac{1}{2}, \quad for\ n > 10$$

For $n < 11$,we can calculate $\lambda_n$numerically, therefore one gets

$$\Phi(u,v) \approx \sum_{n=1}^{10} a_n \exp(-\lambda_n v)\cos\lambda_n u + \sum_{n=11}^{\infty} a_n \exp[-(n-\frac{1}{2})v]\cos(n-\frac{1}{2})u$$

at this point, we need to calculate a_n, by considering the last boundary condition one has

$$\Phi_v(u,0) = f(u) = \sum_{n=0}^{\infty} -\lambda_n a_n \cos\lambda_n u, \qquad 0 < u < \pi$$

using the relation $\tan\lambda_n\pi = \frac{\lambda_n}{h}$,

$$f(u) = \sum_{n=0}^{\infty} -h a_n \tan(\lambda_n\pi)\cos(\lambda_n u), \qquad (1-4)$$

from (1 - 4) we obtain a_n as following

$$a_n = \frac{2}{\pi}\int_0^{\pi} h^{-1}\cot(\lambda_n\pi) f(\xi)\cos\lambda_n\xi d\xi. \qquad (1-5)$$

$$\Phi(u,v) \approx \sum_{n=1}^{10} a_n \exp(-\lambda_n v)\cos\lambda_n u + \sum_{n=11}^{\infty} a_n \exp[-(n-\frac{1}{2})v]\cos(n-\frac{1}{2}).$$

with a_n as , λ_n for $n = 1, 2, \cdots, 10$ can be calculated as follows

Approximate Solution to the equation, $\tan\lambda_n\pi = \frac{\lambda_n}{h}$.

That equation can be re-written as follows,

$\pi h \tan x = x$ with $x = \lambda_n\pi$ or simply, $\pi h \sin x - x\cos x = 0$.

The first root is at $x = 0$, and the interval $(n\pi, n\pi + \frac{\pi}{2})$ contains the other roots .

We would like to find an expression for the zero x_n,

$$(n-1)\pi < x < n\pi - \frac{\pi}{2} \qquad n = 1, 2, 3, \cdots$$

and Newton's iteration formula is. $X_{n+1} = X_n - \frac{f'(X_n)}{f(X_n)} \quad n = 0, 1, 2, \cdots$.

Let $x_{(0)} = n\pi - \frac{\pi}{2}$ and following Newton's iteration formula yields, $g(x) = \pi h \sin x - x\cos x$ and $g'(x) = \pi h \cos x - \cos x + x\sin x$.

$$x_{(1)} = x_{(0)} - \frac{\pi h \sin x_{(0)} - x_{(0)}\cos x_{(0)}}{\pi h \cos x_{(0)} - \cos x_{(0)} + x_{(0)}\sin x_{(0)}}$$

$$x_{(1)} = (n\pi - \frac{\pi}{2}) - [\pi\frac{h}{n\pi - \frac{\pi}{2}}]$$

or simply, $x_{(1)} = (n - \frac{1}{2})\pi - \pi\frac{h}{n-\frac{1}{2}}$.

By continuing the procedure , we obtain $x_{(2)}$ as follows ;

$$g[x_{(1)}] = \pi h \sin x_{(1)} - x_{(1)}\cos x_{(1)}$$

$$g[x_{(1)}] = \pi h \sin[(n-\frac{1}{2})\pi - \frac{h}{n-\frac{1}{2}}] - [(n-\frac{1}{2})\pi - \frac{h}{n-\frac{1}{2}}]\cos[(n-\frac{1}{2})\pi - \frac{h}{n-\frac{1}{2}}]$$

$$g'[x_{(1)}] = \pi h \cos x_{(1)} - \cos x_{(1)} + x_{(1)} \sin x_{(1)}$$

$$g'[x_{(1)}] = \pi h \cos[(n-\frac{1}{2})\pi - \frac{h}{n-\frac{1}{2}}] \quad - \cos[(n-\frac{1}{2})\pi - \frac{h}{n-\frac{1}{2}}] + [(n-\frac{1}{2})\pi - \frac{h}{n-\frac{1}{2}}]\sin[(n-\frac{1}{2})\pi - \frac{h}{n-\frac{1}{2}}]$$

After simplifying and neglecting small terms , we get the following relation for$x_{(2)}$, which seems to be a very good approximation,

$$x_{(2)} = \{(n-\frac{1}{2})\pi - \frac{h}{n-\frac{1}{2}}\} - \frac{h^2}{2}\{(n-\frac{1}{2})^3\pi - 2h(n-\frac{1}{2})\}^{-1}.$$

We may use the first term as an approximate value of $x_(2)$,

$$x_{(2)} \approx \{(n-\frac{1}{2})\pi - \frac{h}{n-\frac{1}{2}}\},$$

which gives the approximation to λ_n as follows;

$$\lambda_n \approx \{(n-\frac{1}{2}) - \frac{h}{(n-\frac{1}{2})}\pi\} \qquad n = 1, 2, 3, \cdots$$

if we choose $h = \frac{1}{4}$ and $f(\xi) = 1$ then we have

$$a_n = \frac{8}{\pi}\cot(\lambda_n \pi)\int_0^{\pi} \cos \lambda_n \xi d\xi = \frac{8}{\pi}\cos(\lambda_n \pi) \qquad n = 1, 2, 3 \cdots .$$

λ_1	λ_2	λ_3	λ_4	λ_5
0.34084505	1.48673708	2.46816901	3.47726358	4.48231611

λ_6	λ_7	λ_8	λ_9	λ_{10}
5.48553136	6.48775731	7.48938967	8.49063794	9.49162342

a_1	a_2	a_3	a_4	a_5
-1.22084704	0.10072636	-0.25422368	0.18173664	-0.14139882

a_6	a_7	a_8	a_9	a_{10}
0.11569890	0.09917640	0.08486600	-0.07996856	0.06700496

and for $n = 20$, $\lambda_{20} = 19.4959191$ and $a_{20} = 0.032646304$.

Note:Even if we use two terms of $x_{(2)}$, the difference is not important, for example, we obtain $\lambda_{10} = 9.49161972 \quad , \quad a_{10} = 0.06703448$.

Case 2: In boundary condition (1) , if we choose h very small

$$(for\ example, \quad h = 0.0001),$$

then the term $h\Phi(0, v)$ is negligible, therefore we obtain approximate solution to problem (1). [1,2,3]

$$\lambda_n \approx \{(n - \frac{1}{2}) - \frac{h}{n - \frac{1}{2}}\pi\} \qquad n = 1, 2, 3, \cdots$$

with $h = 0.0001$, $f(\xi) = 1$.

$$a_n = \frac{2}{\pi}\int_0^{\pi} h^{-1}\cot(\lambda_n\pi)f(\xi)\cos\lambda_n\xi d\xi = \frac{2}{\pi}h^{-1}\cos\lambda_n\pi.$$

λ_1	λ_2	λ_3	λ_4	λ_5
0.49993633	1.49997877	2.49998726	3.49999090	4.49999292

a_1	a_2	a_3	a_4	a_5
1.27339999	-4.24599999	2.54799999	-1.81999990	1.41599998

and for $n = 10$, $\lambda_{10} = 9.49999664 \quad , \quad a_{10} = -6.72 \times 10^{-2}$.

Problem 0.2. *Find the solution to the following M.B.V.P.*
P.D.E. $\Delta\Phi = 0, \qquad x > 0 \ , \ y > 0.$
B.C.

$$\begin{array}{lll} 1) & \Phi_x(0, y) = 0, & y > 0 \\ 2) & \Phi(x, 0) = 0, & 0 < x < a \\ 3) & \Phi_y(x, 0) = f(x), & a < x < 1 \\ 4) & \Phi(x, 0) = 0, & x > 1. \end{array}$$

Solution : Our strategy is to create a sequence of tractable boundary value problems leading to standard integral equations which may be solved analytically. Let $\Phi(x,y)$ be the desired function, then $\Phi(x,y)$ has the following integral representation

$$\Phi(x,y)=\int_0^\infty \Psi(\xi)\exp(-y\xi)\cos(x\xi)d\xi. \qquad (2-1)$$

Using boundary conditions (2), (3), (4) we get the following triple integral equations.

$$\Phi(x,y)=0 \Longrightarrow \int_0^\infty \Psi(\xi)\cos(x\xi)d\xi=0, \quad 0<x<a. \qquad (2-2)$$

$$\Phi_y(x,0)=f(x) \Longrightarrow \int_0^\infty \xi\Psi(\xi)\cos(x\xi)d\xi=-f(x) \quad a<x<1. \qquad (2-3)$$

$$\Phi(x,0)=0 \Longrightarrow \int_0^\infty \Psi(\xi)\cos(x\xi)d\xi=0, \quad 1<x. \qquad (2-4)$$

Now, we introduce the function

$$\Psi(\xi)=\frac{1}{\xi}\int_a^1 h(t^2)\sin(t\xi)dt \qquad (2-5)$$

with subsidiary condition

$$\int_a^1 h(t^2)dt=0 \qquad (2-6)$$

where h(t) is a new unknown function to be determined, we can show that (2-5) satisfies first and third equation, by substituting (2-5) in (2-3) we get the following

$$\int_0^\infty \left[\int_a^1 h(t^2)\sin t\xi d\xi\right]\cos(x\xi)d\xi=-f(x) \qquad (2-7)$$

by changing the order of integration, one gets

$$\int_1^a h(t^2)dt\int_0^\infty \sin(t\xi)\cos(x\xi)d\xi=-f(x) \qquad (2-8)$$

the inner integral has the value $\frac{t}{t^2-x^2}$ in cesaro sense [4], therefore we get the following singular integral equation,

$$\int_a^1 \frac{th(t^2)}{t^2-x^2}dt=-f(x) \qquad (2-9)$$

the above integral equation has the following solution [5], the above integral equation has the following solution [4],

$$h(t^2)=\frac{1}{\pi^2\sqrt{(t^2-a^2)(1-t^2)}}\{\pi\int_{a^2}^1 h(u)du$$
$$+2P.V.\int_a^1 \frac{\sqrt{(u^2-a^2)(1-u^2)}}{t^2-u^2}f(u)udu\} \qquad (2-10)$$

upon substitution (2-10) in (2-5) we obtain

$$\Psi(\xi) = \frac{1}{\xi}\int_a^1 h(t^2)\sin(t\xi)dt$$

and

$$\Phi(x,y) = \int_0^\infty \Psi(\xi)\exp(-y\xi)\cos(x\xi)d\xi.$$

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