Int. J. Contemp. Math. Sci., Vol. 1, 2006, no. 6, 297 - 303

The Hájeck-Rènyi inequality for weighted sums of negatively orthant dependent random variables

Hyun-Chull Kim

Department of Mathematics Education.
Daebul University, Chonnam 526-720, Korea
E-mail : kimhc@mail.daebul.ac.kr

Abstract

In this paper we establish the Hájeck-Rènyi inequality for negatively orthant dependent random variables and derive a strong law of large numbers for weighted sums of these variables by applying this inequality.

Mathematics Subject Classification: 60F05 60F15

Keywords: Negatively orthant dependent random variables; Hájeck-Rènyi inequality; Strong law of large numbers; Rademacher -Mensov inequality; Weighted sum.

1 Introduction

The history and literature on the strong laws of large numbers is vast and rich as this concept is crucial in probability and statistical theory. The literature on concepts of negative dependence is much more limited but still very interesting. Lehmann(1966) provided an extensive introductory overview of various concepts of positive and negative dependence in the bivariate case. Negative dependence has been particularly useful in obtaining strong laws of large numbers(see [7], [10], [11] and [13]).

Hájeck-Rènyi(1955) proved the following important inequality:
If $\{X_n, n \geq 1\}$ is a sequence of independent random variables with $EX_n = 0$ and $EX_n^2 < \infty$, $n \geq 1$, and $\{b_n, n \geq 1\}$ is a positive nondecreasing real sequence, then for any $\epsilon > 0$, any positive integer $m < n$,

$$P(\max_{m \leq k \leq n} |\frac{\sum_{j=1}^{k} X_j}{b_k}| \geq \epsilon) \leq \epsilon^{-2}(\sum_{j=m+1}^{n} \frac{EX_j^2}{b_j^2} + \sum_{j=1}^{m} \frac{EX_j^2}{b_m^2}) \quad (1)$$

$(see[6] \quad and \quad [12])$.

Since then this inequality has been studied by many authors(see [2],[4] and [5]).

In section 2 we introduce the concepts of negatively orthant dependent random variable, and consider some preliminary results. In Section 3 we derive the Hájeck-Rènyi inequality for negatively orthant dependent the random variable and the generalized strong law of large numbers and apply this inequality to obtain a strong law of large number for sum of negatively orthant dependent sequence which have not been established previously in the literature.

2 Preliminaries

This section will contain some background materials on negative orthant dependence which will be used in obtaining the major strong law of large numbers in the next section.

Definition 2.1(Lehmann (1966)) Random variables X and Y are negatively quadrant dependent(NQD) if

$$P\{X \le x, Y \le y\} \le P\{X \le x\}P\{Y \le y\} \tag{2}$$

for all $x, y \in R$. A collection $\{X_n, n \ge 1\}$ of random variables is said to be pairwise NQD if every pair of random variables in the collection satisfies (2).

It is important to note that Definition 2.1 implies

$$P\{X > x, Y > y\} \le P\{X > x\}P\{Y > y\} \tag{3}$$

for all $x, y \in R$. Moreover, it follows that (3) implies (2), and hence, they are equivalent for pairwise NQD. Ebrahimi and Ghosh(1981) showed that (2) and (3) are not equivalent for $n \ge 3$. Consequently, the following definition is needed to define sequences of negatively dependent random variables.

Definition 2.2(Ebrahimi, Ghosh(1981))The random variables $X_1, X_2, \cdots, X_n$ are said to be

(a) lower negative orthant dependent(LNOD) if for each n

$$P\{X_1 \le x_1, \cdots, X_n \le x_n\} \le \prod_{i=1}^{n} P\{X_i \le x_i\} \tag{4}$$

for all $x_1, \cdots, x_n \in R$,

(b) upper negatively orthant dependent(UNOD) if for each n

$$P\{X_1 > x_1, \cdots, X_n > x_n\} \le \prod_{i=1}^{n} P\{X_i > x_i\} \tag{5}$$

for all $x_1, \cdots, x_n \in R$,
(c) negatively orthant dependent(NOD) if both (4) and (5) hold.

The following properties are listed for reference in obtaining the main results in the next section. Detailed proofs can be found in [3] and [9](see [13]).
Lemma 2.3 If $\{X_n, n \geq 1\}$ is a sequence of NOD random variables and $\{f_n, n \geq 1\}$ is a sequence of Borel functions all of which are monotone increasing (or all monotone decreasing), then $\{f_n(X_n), n \geq 1\}$ is a sequence of NOD random variables.

The following lemma is obtained by using an extension of the well-known Rademacher-Mensov inequality((see Lemma 3 of Chandra and Ghosal (1996)).
Lemma 2.4 Let $X_1, \cdots, X_n$ be NOD random variables with $EX_n = 0$ and $EX_n^2 < \infty$ for all $n \geq 1$. Then we have

$$E(X_{m+1} + \cdots + X_{m+p})^2 \leq EX_{m+1}^2 + \cdots + EX_{m+p}^2 \tag{6}$$

for all $m, p \geq 1,\ m + p \leq n$. Morever, we have

$$E(\max_{1 \leq k \leq n} (\sum_{i=1}^{k} X_i)^2) \leq ((\log n / \log 3) + 2)^2 \sum_{i=1}^{n} EX_i^2. \tag{7}$$

3 Main results

By using Lemma 2.4 we obtain the following Hàjeck-Rényi type inequality for NOD random variables:
Theorem 3.1 Let $\{b_n, n \geq 1\}$ be a positive sequence of nondecreasing real numbers. Let $\{X_n, n \geq 1\}$ be a sequence of NOD random variables with $EX_n = 0$ and $EX_n^2 < \infty$. Then

$$P\{\max_{1 \leq k \leq n} |\frac{\sum_{i=1}^{k} X_i}{b_k}| \geq \epsilon\} \leq 8\epsilon^{-2}((\log n / \log 3) + 2)^2 \sum_{i=1}^{n} \frac{EX_i^2}{b_i}. \tag{8}$$

Proof Without loss of generality, setting $b_0 = 0$, we have

$$\begin{aligned} S_k &= \sum_{j=1}^{k} b_j \frac{X_j}{b_j} \\ &= \sum_{j=1}^{k} (\sum_{i=1}^{j} (b_i - b_{i-1}) \frac{X_j}{b_j}) \\ &= \sum_{i=1}^{k} (b_i - b_{i-1}) \sum_{i \leq j \leq k} \frac{X_j}{b_j}. \end{aligned}$$

Note that $(1/b_k)\sum_{j=1}^{k}(b_j - b_{j-1}) = 1$. So

$$\{|\frac{S_k}{b_k}| \geq \epsilon\} \subset \{\max_{1\leq i\leq k} |\sum_{i\leq j\leq k} \frac{X_j}{b_j}| \geq \epsilon\}.$$

Therefore,

$$\begin{aligned}\{\max_{1\leq k\leq n} |\frac{S_k}{b_k}| \geq \epsilon\} &\subset \{\max_{1\leq k\leq n}\max_{1\leq i\leq k} |\sum_{i\leq j\leq k} \frac{X_j}{b_j}| \geq \epsilon\} \\ &= \max_{1\leq i\leq k\leq n} |\sum_{j\leq k} \frac{X_j}{b_j} - \sum_{j<i} \frac{X_i}{b_j}| \geq \epsilon\} \\ &\subset \{\max_{1\leq i\leq n} |\sum_{j=1}^{i} \frac{X_j}{b_j}| \geq \frac{\epsilon}{2}\}.\end{aligned}$$

By Lemma 2.4 we obtain the following result

$$P\{\max_{1\leq k\leq n} |\frac{S_k}{b_k} \geq \epsilon\} \leq 8\epsilon^{-2}((\log n/\log 3)+2)^2 \sum_{i=1}^{n} \frac{EX_i^2}{b_i^2}.$$

From Theorem 3.1 we can get the following more generalized Hàjeck-Rényi inequality.

Corollary 3.2 Let $\{b_n, n \geq 1\}$ be a positive sequence of nondecreasing real numbers. Let $\{X_n, n \geq 1\}$ be a sequence of NOD random variables with $EX_n = 0$ and $EX_n^2 < \infty$. Then, for all $\epsilon > 0$ and any positive integer $m < n$, we have

$$P\{\max_{m\leq k\leq n} |\frac{\sum_{i=1}^{k} X_i}{b_k}| \geq \epsilon\}$$

$$\leq 32\epsilon^{-2}((\log n/\log 3)+2)^2 (\sum_{j=m+1}^{n} \frac{EX_j^2}{b_j^2} + \sum_{j=1}^{m} \frac{EX_j^2}{b_m^2}). \tag{9}$$

Now we discuss the main strong law of large numbers by applying Corollary 3.2.

Theorem 3.3 Let $\{b_n, n \geq 1\}$ be a positive sequence of nondecreasing real numbers. Let $\{X_n, n \geq 1\}$ be a sequence of NOD random variables with $EX_n = 0$ and $EX_n^2 < \infty$. Assume

$$\sum_{n=1}^{\infty} \frac{EX_n^2}{b_n^2} < \infty. \tag{10}$$

Then,

(A) For any $0 < r < 2$, $E\sup_n(|S_n| \,/\, b_n \log n)^r < \infty$,

(B) $0 < b_n \uparrow \infty$ implies $S_n/b_n \log n \to 0$ *a.s.*, where $S_n = \sum_{i=1}^{n} X_i,\ n \geq 1$.

Proof (A) Note that, for any $0 < r < 2$

$$E\sup_n\left(\frac{|S_n|}{b_n \log n}\right)^r < \infty \iff \int_1^\infty P\{\sup_n \frac{|S_n|}{b_n \log n} > t^{\frac{1}{r}}\}dt < \infty.$$

By Corollary 3.2 it follows from (10) that

$$\begin{aligned}
&\int_1^\infty P\{\sup_n \frac{|S_n|}{b_n \log n} > t^{\frac{1}{r}}\}dt \\
&\leq 64 \int_1^\infty t^{-\frac{2}{r}} \lim_{n\to\infty} (\log n)^{-2}((\log n/\log 3) + 2)^2 \\
&\times (\sum_{j=m+1}^{n} \frac{EX_j^2}{b_j^2} + \sum_{j=1}^{m} \frac{EX_j^2}{b_m^2})dt \\
&= 64 \lim_{n\to\infty} (\log n)^{-2}((\log n/\log 3) + 2)^2 \\
&\times (\sum_{j=m+1}^{n} \frac{EX_j^2}{b_j^2} + \sum_{j=1}^{m} \frac{EX_j^2}{b_m^2}) \int_1^\infty t^{-\frac{2}{r}} dt \\
&< \infty.
\end{aligned}$$

Hence the proof (A) is complete.

(B) By Corollary 3.2 we have

$$\begin{aligned}
&P\{\max_{m\leq k\leq n} |\frac{\sum_{i=1}^{k} X_i}{b_k}| \geq \epsilon \log n\} \\
&\leq 32\epsilon^{-2}(\log n)^{-2}((\log n/\log 3) + 2)^2(\sum_{j=m+1}^{n} \frac{EX_j^2}{b_j^2} + \sum_{j=1}^{m} \frac{EX_j^2}{b_m^2}).
\end{aligned}$$

But

$$\begin{aligned}
&P\{\sup_n |\frac{\sum_{i=1}^{n} X_i}{b_n}| \geq \epsilon \log n\} \\
&= \lim_{n\to\infty} P\{\max_{m\leq k\leq n} |\frac{\sum_{i=1}^{k} X_i}{b_k}| \geq \epsilon \log n\} \\
&\leq \lim_{n\to\infty} 32\epsilon^{-2}(\log n)^{-2}((\log n/\log 3) + 2)^2 \\
&\times (\sum_{j=m+1}^{n} \frac{EX_j^2}{b_j^2} + \sum_{j=1}^{m} \frac{EX_j^2}{b_m^2}). \qquad (11)
\end{aligned}$$

By the Kronecker lemma it follows from (10) that

$$\sum_{j=1}^{m} \frac{EX_j^2}{b_m^2} \to 0 \quad as\ m \to \infty. \qquad (12)$$

Hence, combining (10)-(12) yields

$$\lim_{n\to\infty} P\{\sup_{k<=n} |\frac{\sum_{i=1}^{k} X_i}{b_k}| \geq \epsilon \log n\} = 0, \tag{13}$$

which completes the proof of (B) of Theorem 3.3.

Corollary 3.4 Let $\{X_n, n \geq 1\}$ be a sequence of NOD random variables with $EX_n = 0$ and $EX_n^2 < \infty$. Assume

$$\sup_n EX_n^2 < \infty.$$

Then, for $0 < t < 2$ and for all $\epsilon > 0,\ m \geq 1$

$$P\{\sup_{n\geq m} |\sum_{i=1}^{k} X_i|/n^{\frac{1}{t}} \geq \epsilon \log n\}$$
$$\leq 32\epsilon^{-2}(\log n)^{-2}((\log n/\log 3)+2)^2 \frac{2}{2-t} \sup_n EX_n^2 m^{(t-2)/t}.$$

From Theorem 3.3 we prove the following strong law of large numbers for NOD random variables with $EX_n = 0$ and $EX_n^2 < \infty$.

Corollary 3.5 Let $\{X_n, n \geq 1\}$ be a sequence of NOD random variables with $EX_n = 0$ and $EX_n^2 < \infty$. Assume

$$\sum_{n=1}^{\infty} EX_n^2 < \infty.$$

Then, for $0 < t < 2$, $S_n/n^{\frac{1}{t}} \log n \to 0$ a.s. as $n \to \infty$ where $S_n = \sum_{i=1}^{n} X_i$.

Corollary 3.6 Let $\{X_n, n \geq 1\}$ be a sequence of NOD random variables with $EX_n = 0$ and $EX_n^2 < \infty$. Assume $\sup_n EX_n^2 < \infty$. Then, for $0 < t < 2$

$$(A)\ S_n/n^{\frac{1}{t}} \log n \to 0 \ \ a.s, \ \ as \ \ n \to \infty$$
$$(B)\ E\sup_n(|S_n| \ / \ n^{\frac{1}{t}} \log n)^r < \infty \ \ for\ any \ \ 0 < r < 2.$$

References

[1] Chandra, T.K., Ghosal, S., 1996, *Extensions of the strong law of large numbers of Marcinkiewicz and Zygmund for dependent random variables,* Acta. Math. Hung. **71**, 327-336

[2] Christofides, T. C., 2000, *Maximal inequalities for demimartingales and a strong law of large numbers*, Statist. Probab. Lett. **50**, 357-363

[3] Ebrahimi, N., Ghosh, M., 1981, *Multivariate Negative Dependence*, Commun. Stat. Theory and Methods A **10**, 307-336

[4] Gan, S., 1997, *The Hájeck-Rènyi inequality for Banach space valued rnadom variables and its application*, Wuhan Uni. J. Natural Sci. **2**, 13-18

[5] Gan, S. 1997, *The Hájeck-Rènyi inequality for Banach space valued martingales and the p-smoothness of Banach space*, Statist. Probab. Lett. **32** 245-248

[6] Hájeck-Rènyi, A., 1955, *Generalization of an inequality of Kolmogorov.* Acta. Math. Acad. Sci. Hungar. **6** 281-283

[7] Kim, T.S. and Back, J.I, 1999, *The strong laws of large numbers for weighted sums of pairwise quadrant dependent random variables*, J. Korean Math. Soc. **36** 37-49

[8] Kim, T.S., Ko, M.H. and Lee, I.H., 2002, *On the strong law of large numbers for asymptotically almost negatively associated random variables*, Rocky Mountaln J. Math. To appear.

[9] Lehmann, E.L., 1966, *Some Concepts of Dependence*, Ann. Math. Stat. **43**, 1137-1153

[10] Matula, P., 1992, *A note on the almost sure convergence of sums of negatively dependent random variables*, Statist. Probab. Lett. **15**, 209-213

[11] Peligrad, M. and Gut, A., 1999, *Almost sure results for a class of dependent random variables*, J. Theor. Probab. **12**, 87-104

[12] Stout, W. F., 1974, *Almost Sure Convergence*, Academic Press, New York.

[13] Taylor, R.L., Patterson, R.F. and Bozorgnica, A., 2002, *A strong law of large numbers for arrays of rowwise negatively dependent random variables*. Stoch. Anal. Appl. **20** 643-656

Received: October 31, 2005