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Contraction Mapping on Probabilistic Hyperspace and an Application

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Abstract

We considered a class of precompact subsets of a PM space (here we called probabilistic hyperspace) and applied the fixed point principle in this space to obtain an attractor of the dynamical system defined through the iterated function systems (IFS).

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1 Introduction

Let $\overline{\mathbf{R}} = [-\infty, \infty]$ be the extended real line. A *distance distribution function* (d.d.f) F is a non-decreasing function from $\overline{\mathbf{R}}^+ = [0, +\infty]$ into $[0, 1]$, which is left continuous on $(0, \infty)$ and takes on the values $F(0) = 0$ and $F(+\infty) = 1$. The set of all d.d.f. is denoted by Δ^+. We shall also consider the subset $D^+ = \{F \in \Delta^+ | \lim_{x\to\infty} F(x) = 1\} \subset \Delta^+$.

For $a < \infty$, the element $\varepsilon_a \in \Delta^+$ is defined as

$$\varepsilon_a(x) = \begin{cases} 0, & t \leq a; \\ 1, & t > a. \end{cases}$$

By setting $F \leq G$ whenever $F(x) \leq G(x)$, $\forall x \in \mathbf{R}^+$, one introduces a natural ordering in Δ^+. The maximal element for Δ^+ in this ordering is the d.d.f ε_0

and the minimal element is ε_∞. Convergence in Δ^+ is assumed to be weakly convergence, i.e. $F_n \to F$ if and only if $F_n(x) \to F(x)$ at each continuity point x of F.

A *triangle function* is a mapping $\tau\colon \Delta^+ \times \Delta^+ \to \Delta^+$ that is commutative, associative, nondecreasing in each variable and has ε_0 as identity. A triangle function τ is said to be sup-continuous if for every family $\{F_\lambda | \lambda \in \Lambda\}$ of d.d.f.'s in Δ^+ and every $G \in \Delta^+$:

$$\sup_{\lambda\in\Lambda} \tau(F_\lambda, G) = \tau\left(\sup_{\lambda\in\Lambda} F_\lambda, G\right).$$

The typical triangle function is the operation τ_T which is given by

$$\tau_T(F,G)(x) = \sup_{u+v=x} T(F(u), G(v)),$$

for all $F, G \in \Delta^+$ and all $x > 0$ [[4], Secs. 7.2 and 7.3]. Here T is a *t-norm*, i.e., T is binary operation on $[0,1]$ that is commutative, associative, nondecreasing in each place and has 1 as identity. The most important t-norms are the function W, $Prod$, and M which are defined, respectively, by

$$\begin{aligned} W(a,b) &= \max\{a+b-1, 0\}, \\ Prod(a,b) &= ab, \\ M(a,b) &= \min(a,b). \end{aligned}$$

Let S be a nonempty set and τ be a triangle function on Δ^+.

Definition 1 *A mapping $F\colon S \times S \to \Delta^+$ is called a probabilistic metric on S if F satisfies the conditions: $\forall p, q, r \in S$,*
(PM1) $F_{pq} = \varepsilon_0 \Leftrightarrow p = q$,
(PM2) $F_{pq} = F_{qp}$,
(PM3) $F_{pr} \geq \tau(F_{pq}, F_{qr})$.
The triple (S, F, τ) is called a probabilistic metric space (in short PM space).

The detailed study of PM spaces can be found in [4].

Definition 2 *Let (S, F, τ) be a PM space and A is a nonempty subset of S.*

1. *A is said to be closed if and only if $F_{pA} \neq \varepsilon_0$ whenever $p \notin A$, where $F_{pA}(x) = \sup_{q\in A} F_{pq}(x), \forall x > 0$ is the probabilistic distance between a point p and a set A.*

2. *The strong closure of A in the sense of Kuratowski [4] is defined as*

$$\overline{A} = \{p \in S | F_{pA} = \varepsilon_0\}.$$

3. *A is said to be bounded if* $\lim_{x\to+\infty} D_A(x) = 1$, *where* $D_A(x) = \sup_{t<x} \inf_{p,q\in A} F_{pq}(t)$ *is the probabilistic diameter of* A. *Since for any* $p, q \in A, F_{pq}(x) \geq D_A(x)$, *[[4], Theorem 12.4.2], this implies that* $\lim_{x\to+\infty} F_{pq}(x) = 1$ *for every* $p, q \in A$.

The notion of precompact was introduced by Istratescu [3] as follows:

Definition 3 ([3]) *Let* (S, F, τ) *be a PM space and* A *be a bounded subset of* S. *The set* A *is said to be precompact if for any* $t > 0$, *there exists a finite cover* $\{A_1, A_2, \ldots, A_n\}$ *of* A *such that*

$$D_{A_j}(t) > 1 - t, \quad \forall j = 1, 2, \ldots, n.$$

Theorem 1 ([3]) *A PM space* (S, F, τ) *is said to be precompact iff for every* $t > 0$, *there exists a finite set* A_t *in* S *such that for any* $p \in S$, *there is a* $q \in A_t$ *such that*

$$F_{pq}(t) > 1 - t.$$

Sehgal and Bharucha [5] introduced the notion of a contraction mapping in PM spaces and provided a prove for a fixed point theorem in complete PM spaces. Further work on contraction mappings and fixed point theorem in PM spaces can be found in [[2],[3]].

Definition 4 ([5]) *Let* (S, F, τ) *be a PM space. A mapping* $f : S \to S$ *is called a probabilistic contraction map on* (S, F, τ) *if there exists a* $k \in (0, 1)$, *called a contractive factor such that for every* $p, q \in S$ *and* $x > 0$,

$$F_{f(p),f(q)}(x) \geq F_{pq}\left(\frac{x}{k}\right).$$

Theorem 2 ([5]) *Let* (S, F, τ) *be a complete PM space. If* $f : S \to S$ *is a probabilistic contraction mapping on* (S, F, τ) *then* f *posses exactly one fixed point* $p \in S$, *i.e.,* $f(p) = p$.

2 Probabilistic Hyperspaces

Let (S, F, τ) be a PM space and
$K_F(S) = \{A \subset S | A$ is nonempty and precompact set$\}$. For any $A, B \in K_F(S)$, the *probabilistic Hausdorff distance* between A and B [2] is given by

$$H_{AB}(x) = \mathrm{Min} \sup_{t<x} \left\{ \inf_{p\in A} F_{pB}(t), \inf_{q\in B} F_{Aq}(t) \right\}, \tag{1}$$

where $F_{pA}(x) = \sup_{q\in A} F_{pq}(x)$.

From [[4], Theorem 12.9.3, 12.9.4 and 12.9.5], H_{AB} is a probabilistic metric provided τ is sup-continuous in (S, F, τ). Whence we conclude that the triple $(K_F(S), H, \tau)$ is a PM space with τ is sup-continuous in (S, F, τ). We shall call the complete PM space $(K_F(S), H, \tau)$ as a *probabilistic hyperspace*.

We prove the following fixed point theorem which will be the extension of Theorem 2.

Theorem 3 *Let (S, F, τ) be a bounded complete PM space with Range$(F) \subset D^+$. Let $T \colon K_F(S) \to K_F(S)$ (where the set $K_F(S)$ as defined above) be a mapping such that*

$$H_{TA,TB}(x) \geq H_{AB}\left(\frac{x}{k}\right), \tag{2}$$

for every $A, B \in K_F(S)$ and for every $k \in (0, 1)$. Then T posses an unique fixed point $U \in K_F(S)$.

Proof From (1), the left side of (2) can be expressed as:

$$H_{TA,TB}(x) \geq \mathrm{Min} \sup_{t<x} \left\{ \inf_{p\in A} F_{Tp,TB}(t), \inf_{q\in B} F_{TA,Tq}(t) \right\}.$$

Hence, for every $Tp \in TA$, there exists $Tq \in TB$ so that,

$$H_{TA,TB}(x) \geq \sup_{t<x} F_{Tp,Tq}(t). \tag{3}$$

By using (1) again to the right side of (2) and we can conclude that, there exists $k \in (0, 1)$ such that for every $p \in A$, there exist $q \in B$, so that

$$F_{AB}(x) \geq \sup_{kt<x} F_{pq}(t). \tag{4}$$

By comparing both (3) and (4) with (2), one can conclude that:

$$\sup_{t<x} F_{Tp,Tq}(t) \geq \sup_{kt<x} F_{pq}(t). \tag{5}$$

Let $y = kt > 0$. Then by noting that $x > t > kt = y$, we write (5) as:

$$\sup_{y<x} F_{Tp,Tq}(y) \geq \sup_{y<x} F_{pq}\left(\frac{y}{k}\right),$$

which implies that

$$F_{Tp,Tq}(y) \geq F_{pq}\left(\frac{y}{k}\right), \tag{6}$$

for every $y > 0$ and $k \in (0,1)$.

It is worth noting that, if for each $p, q \in U \subset S$ with $\overline{U} \in K_F(S)$, the mapping $T\colon \overline{U} \to \overline{U}$ is a contraction mapping in (S, F, τ) which satisfies the condition (6). Then by Theorem 2, T has a unique fixed point, say $u \in \overline{U}$. Since $u \in \overline{U}$ is arbitrary in $\overline{U}$, we can conclude that the mapping $T\colon K_F(S) \to K_F(S)$ has unique fixed point $\overline{U} \in K_F(S)$.

3 Application

In this section we consider the iterated function system (IFS) on a precompact PM space which is a generalization of IFS on a complete metric spaces (Barnsley [1]).

Definition 5 *An iterated function system (IFS) on a precompact PM space* (S, F, τ) *is a finite collection of contraction mappings* $f_i : S \to S, i = 1, \ldots, n$, *with the contractive property: for every* $p, q \in S$ *and* $x > 0$,

$$F_{f_i(p), f_i(q)}(x) = F_{p,q}\left(\frac{x}{k_i}\right)$$

where $k_i, i = 1, \ldots, n$, *is the contractivity factors.*

Our goal is to describe the *attractor* of an IFS. The basic step in forming this attractor is to start with a precompact set $A \in K_F(S)$ and apply the contractions $f_i(A), i = 1, \ldots, n$. We can formulate this process as a mapping of precompact sets:

$$W(A) = \bigcup_{i=1}^{n} f_i(A) \tag{7}$$

and a dynamical system

$$A \mapsto W(A) \mapsto W^2(A) \mapsto \cdots \mapsto W^m(A) \mapsto \cdots$$

where the attractor is the limit of this sequence $\{W^m(A)\}$.

Let $(K_F(S), H, \tau)$ be a probabilistic hyperspace as defined in section 2. We say that a sequence $\{A_n\} \subset K_F(S)$ converges weakly to $A \in K_F(S)$ if $F_{A_nA} \to \varepsilon_0$ as $n \to \infty$.

Returning to the IFS $\{f_1, \ldots, f_n\}$ and the associated map $W : K_F(S) \to K_F(S)$ as defined in (7), we claim that the sequence of iterates $W^m(U)$ will converges weakly to some precompact set U. To prove this, we need the following theorem.

Theorem 4 *Let $(K_F(S), H, \tau)$ be a probabilistic hyperspace. The mapping $W : K_F(S) \to K_F(S)$ is a probabilistic contraction on $K_F(S)$ with the contractivity factor $k \in (0,1)$, i.e.,*

$$H_{W(A),W(B)}(x) \geq H_{AB}\left(\frac{x}{k}\right)$$

for every $A, B \in K_F(S)$.

Proof Let $W(A) = \bigcup_{i=1}^n f_i(A)$ and $W(B) = \bigcup_{i=1}^n f_i(B)$. For any $p \in W(A)$ and $x > 0$,

$$\begin{aligned} F_{p,W(B)}(x) &= \sup\{F_{p,q}(x) | q \in W(B)\} \\ &= \sup\{F_{p,f_i(B)}(x) | i = 1, \ldots, n\} \\ &\geq F_{p,f_j(B)}(x) \end{aligned}$$

for some $j = 1, \ldots, n$. Since $p \in W(A)$, we have $p = f_j(\hat{p})$ for some $j = 1, \ldots, n$ and for some $\hat{p} \in A$. Hence, we have

$$F_{p,W(B)}(x) \geq F_{p,f_j(B)}(x) = F_{f_j(\hat{p}),f_j(B)}(x), \quad x > 0.$$

Since $q = f_j(\hat{q})$, for some $\hat{q} \in B$, by the principle of contraction, we have

$$F_{p,W(B)}(x) \geq F_{f_j(\hat{p}),f_j(\hat{q})}(x) \geq F_{\hat{p},\hat{q}}\left(\frac{x}{k_j}\right), \quad \forall j = 1, 2, \ldots \text{ and } \quad x > 0.$$

By letting $k = \max\{k_j | j = 1, 2, \ldots, n\} < 1$,

$$F_{p,W(B)}(x) \geq F_{\hat{p},\hat{q}}\left(\frac{x}{k}\right), \quad x > 0. \tag{8}$$

Note that, (8) can be written has

$$F_{p,W(B)}(x) \geq \sup_{\hat{q} \in B} F_{\hat{p},\hat{q}}\left(\frac{x}{k}\right) = F_{\hat{p},B}\left(\frac{x}{k}\right), \quad \text{for every } x > 0 \text{ and } k \in (0,1).$$

Since $p \in W(A)$ and $\hat{p} \in A$, one can conclude that:

$$\inf_{p \in W(A)} F_{p,W(B)}(x) \geq \inf_{\hat{p} \in A} F_{\hat{p},B}\left(\frac{x}{k}\right), \quad \text{for every } x > 0 \text{ and } k \in (0,1). \tag{9}$$

By the symmetric property of the probabilistic metric, and with $\hat{q} \in B, q = f_j(\hat{q})$ and $k = \max k_i$, one can show that:

$$\inf_{q \in W(B)} F_{W(A),q}(x) \geq \inf_{\hat{q} \in B} F_{A,\hat{q}}\left(\frac{x}{k}\right), \quad \text{for every } x > 0 \text{ and } k \in (0,1).$$

Thus, by the property of (1), (8) and (9), we have:

$$\begin{aligned} H_{W(A),W(B)}(x) &= \operatorname{Min} \sup_{t<x} \left\{ \inf_{p \in W(A)} F_{p,W(B)}(t), \inf_{q \in W(B)} F_{W(A),q}(t) \right\} \\ &\geq \operatorname{Min} \sup_{t<x} \left\{ \inf_{\hat{p} \in A} F_{\hat{p},B}\left(\frac{t}{k}\right), \inf_{\hat{q} \in B} F_{A,\hat{q}}\left(\frac{t}{k}\right) \right\} \\ &= H_{AB}\left(\frac{x}{k}\right) \end{aligned}$$

Thus, $H_{W(A),W(B)}(x) \geq H_{AB}\left(\frac{x}{k}\right)$ for every $x > 0$ and $k \in (0,1)$. And this completes the proof.

Now, Theorem 3 determines a unique fixed point U of W and asserts that all sequences of iterates of $W^m(U)$ converges upon U. Hence we have the attractor of the IFS, which is the fixed point $U \in K_F(S)$.

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