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FEKETE-SZEGÖ PROBLEM FOR CERTAIN SUBCLASS OF QUASI-CONVEX FUNCTIONS

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Abstract

For $0 \leq \alpha < 1$, let $\mathcal{Q}_\alpha$ be the class of functions f which are normalised analytic and univalent in $\mathcal{D} = \{z : |z| < 1\}$ satisfying the condition

$$Re \left\{ \frac{\alpha(z^2 f''(z))'}{g'(z)} + \frac{(zf'(z))'}{g'(z)} \right\} > 0,$$

where g is a normalised convex function. For $f \in \mathcal{Q}_\alpha$, sharp bounds are obtained for the Feketo-Szegö functional $|a_3 - \mu a_2^2|$ when μ is real.

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1. Introduction

Let $\mathcal{S}$ denote the class of normalised analytic univalent functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

where $z \in \mathcal{D} = \{z : |z| < 1\}$. We also denote by $\mathcal{S}^{\star}$, $\mathcal{C}$ and $\mathcal{K}$ the subclasses of $\mathcal{S}$ consisting of functions which are, respectively, starlike, convex and close-to-convex in $\mathcal{D}$.

A classical result of Feketo and Szegö [2] determines the maximum value of $|a_3 - \mu a_2^2|$, as a function of the real parameter μ, for functions belonging to $\mathcal{S}$. There are now several results of this type in the literature, each of them dealing with $|a_3 - \mu a_2^2|$ for various classes of functions (see, e.g., [1,4]).

Denote by $\mathcal{Q}(\beta)$ the class of strongly quasi-convex functions of order $\beta(\beta \geq 0)$. Thus $f \in \mathcal{Q}(\beta)$ if and only if there exists $f \in \mathcal{C}$ such that for $z \in \mathcal{D}$,

$$\left| arg \quad \left\{ \frac{(zf'(z))'}{g'(z)} \right\} \right| \leq \frac{\pi\beta}{2}.$$

In particular, $\mathcal{Q} = \mathcal{Q}(1)$ is the class of quasi-convex functions introduced by Noor [7]. We also note that every quasi-convex function is close-to-convex and hence univalent in $\mathcal{D}$. For functions belonging to the class $\mathcal{Q}(\beta)$, sharp upper bounds for the functional $|a_3 - \mu a_2^2|$ have been obtained by Nak Eun Cho [6].

In this paper, we give an estimate for the same functional for the class $\mathcal{Q}_\alpha$ defined as follows:

Definition 1 *Let f be given by (1) and $0 \leq \alpha < 1$. Then $f \in \mathcal{Q}_\alpha$ if and only if there exist $g \in \mathcal{C}$ such that for $z \in \mathcal{D}$,*

$$Re \quad \left\{ \frac{\alpha(z^2 f''(z))'}{g'(z)} + \frac{(zf'(z))'}{g'(z)} \right\} > 0. \tag{2}$$

Here, $\mathcal{C}$ denotes the class of convex functions; that is $g \in \mathcal{C}$ if and only if g is analytic in $\mathcal{D}$ and

$$Re \quad \left\{ 1 + \frac{zg''(z)}{g'(z)} \right\} > 0 \tag{3}$$

for $z \in \mathcal{D}$.

We note that by using a lemma due to Miller and Mocanu [5], it can easily be shown that $\mathcal{Q}_\alpha \subset \mathcal{Q}$ for $0 \leq \alpha < 1$ and hence $f \in \mathcal{Q}_\alpha$ means f is univalent.

We first state some preliminary lemmas, required for proving our result.

2. Preliminary Results

Lemma 1 *([8]) Let h be analytic in $\mathcal{D}$ with $Re\ h(z) > 0$ and be given by $h(z) = 1 + c_1 z + c_2 z^2 + ...$ for $z \in \mathcal{D}$, then*

$$\left|c_2 - \frac{c_1^2}{2}\right| \leq 2 - \frac{|c_1|^2}{2}.$$

Lemma 2 *([3]) Let $g \in \mathcal{C}$ with $g(z) = z + b_2 z^2 + b_3 z^3 + \ldots$ Then, for μ real*

$$|b_3 - \mu b_2^2| \leq max \ \left\{\frac{1}{3}, |\mu - 1|\right\}.$$

Lemma 3 *Let $f \in \mathcal{Q}_\alpha$ and be given by (1) then*

$$(\alpha + 1)|a_2| \leq 1$$

and

$$(2\alpha + 1)|a_3| \leq 1.$$

Proof.

Since $g \in \mathcal{C}$, it follows from (3) that

$$g'(z) + zg''(z) = g'(z)p(z) \tag{4}$$

for $z \in \mathcal{D}$, with $Re\ p(z) > 0$ given by $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots$ Equating coefficients, we obtain

$$2b_2 = p_1 \tag{5}$$

and

$$6b_3 = p_2 + 2b_2 p_1. \tag{6}$$

It also follows from (2) that

$$\alpha(z^2 f''(z))' + (zf'(z))' = g'(z)h(z) \tag{7}$$

where $Re\ h(z) > 0$. Writing $h(z) = 1 + c_1 z + c_2 z^2 + \ldots$ and equating coefficients in (7) gives

$$4(\alpha + 1)a_2 = c_1 + 2b_2 \tag{8}$$

and

$$9(2\alpha + 1)a_3 = c_2 + 2b_2 c_1 + 3b_3. \tag{9}$$

The result now follows on using classical inequalities $|p_1| \leq 2,\ |p_2| \leq 2,\ |c_1| \leq 2,\ |c_2| \leq 2$ and the inequalities $|b_2| \leq 1$ and $|b_3| \leq 1$ which follow from (5) and (6).

3. **Main Result**

Theorem. *Let f be given by (1) and belongs to the class $\mathcal{Q}_\alpha$. Then, for $0 \le \alpha < 1$,*

$$9(2\alpha+1)(\alpha+1)^2|a_3-\mu a_2^2|$$
$$\le \begin{cases} 9(\alpha+1)^2-9(2\alpha+1)\mu\ , & \text{if} \quad \mu \le \frac{4(\alpha+1)^2}{9(2\alpha+1)}, \\ 5(\alpha+1)^2-\frac{9(2\alpha+1)\mu}{4}+ \\ \quad \frac{(8(\alpha+1)^2-9(2\alpha+1)\mu)^2}{36(2\alpha+1)\mu}\ , & \text{if} \quad \frac{4(\alpha+1)^2}{9(2\alpha+1)} \le \mu \le \frac{8(\alpha+1)^2}{9(2\alpha+1)}, \\ 3(\alpha+1)^2\ , & \text{if} \quad \frac{8(\alpha+1)^2}{9(2\alpha+1)} \le \mu \le \frac{4(\alpha+1)^2}{3(2\alpha+1)}, \\ -9(\alpha+1)^2+9(2\alpha+1)\mu\ , & \text{if} \quad \mu \ge \frac{4(\alpha+1)^2}{3(2\alpha+1)}. \end{cases}$$

Inequalities are sharp for all cases.

Proof.

From (5),(7),(8) and (9), it is easily established that

$$9(2\alpha+1)(a_3-\mu a_2^2)$$
$$= 3\left\{b_3-\frac{3(2\alpha+1)\mu}{4(\alpha+1)^2}b_2^2\right\}+\left\{c_2+\left(\frac{8(\alpha+1)^2-9(2\alpha+1)\mu}{16(\alpha+1)^2}-\frac{1}{2}\right)c_1^2\right\}$$
$$+\left\{1-\frac{9(2\alpha+1)\mu}{8(\alpha+1)^2}\right\}p_1c_1. \tag{10}$$

First, consider the case $\frac{4(\alpha+1)^2}{9(2\alpha+1)} \le \mu \le \frac{8(\alpha+1)^2}{9(2\alpha+1)}$.

Equation (10) gives

$9(2\alpha+1)|a_3-\mu a_2^2| \le 3\left|b_3-\frac{3(2\alpha+1)\mu}{4(\alpha+1)^2}b_2^2\right|+\left|c_2-\frac{1}{2}c_1^2\right|+\frac{1}{16(\alpha+1)^2}|8(\alpha+1)^2-9(2\alpha+1)\mu||c_1|^2$

$$+\frac{1}{8(\alpha+1)^2}|8(\alpha+1)^2-9(2\alpha+1)\mu||c_1||p_1|$$

$$\le \left(3-\frac{9(2\alpha+1)\mu}{4(\alpha+1)^2}\right)+\left(2-\frac{1}{2}|c_1|^2\right)+\frac{1}{16(\alpha+1)^2}(8(\alpha+1)^2-9(2\alpha+1)\mu)|c_1|^2$$

$$+\frac{1}{4(\alpha+1)^2}(8(\alpha+1)^2-9(2\alpha+1)\mu)|c_1|$$

$$= \varphi(x), \text{ say, with } x = |c_1|,$$

where we have used Lemma 1 and Lemma 2 and the inequality $|p_1| \leq 2$. Elementary calculation indicates that the function φ attains its maximum value at $x_o = \frac{2(8(\alpha+1)^2 - 9(2\alpha+1)\mu)}{9(2\alpha+1)\mu}$ and thus establishing

$$9(2\alpha+1)(\alpha+1)^2|a_3 - \mu a_2^2| \leq \varphi(x_o)$$

$$= 5(\alpha+1)^2 - \frac{9(2\alpha+1)\mu}{4} + \frac{(8(\alpha+1)^2 - 9(2\alpha+1)\mu)^2}{36(2\alpha+1)\mu}\ .$$

Next, since $|x_o| \leq 2$, thus we have $\mu \geq \frac{4(\alpha+1)^2}{9(2\alpha+1)}$ and hence completing the proof for the case $\frac{4(\alpha+1)^2}{9(2\alpha+1)} \leq \mu \leq \frac{8(\alpha+1)^2}{9(2\alpha+1)}$.

Letting $c_1 = \frac{2(8(\alpha+1)^2 - 9(2\alpha+1)\mu)}{9(2\alpha+1)\mu}$, $c_2 = p_1 = p_2 = 2$ and $b_2 = b_3 = 1$ in (10) shows that the result is sharp.

Secondly, we consider the case $\mu \leq \frac{4(\alpha+1)^2}{9(2\alpha+1)}$.

Write

$$a_3 - \mu a_2^2 = a_3 - \frac{4(\alpha+1)^2}{9(2\alpha+1)}a_2^2 + \left(\frac{4(\alpha+1)^2}{9(2\alpha+1)} - \mu\right)a_2^2.$$

Since $|a_2| \leq \frac{1}{\alpha+1}$, it follows that

$$\begin{aligned}9(2\alpha+1)(\alpha+1)^2|a_3 - \mu a_2^2| &\leq 9(2\alpha+1)(\alpha+1)^2\left|a_3 - \tfrac{4(\alpha+1)^2}{9(2\alpha+1)}a_2^2\right| \\ &\quad + 9(2\alpha+1)(\alpha+1)^2\left(\tfrac{4(\alpha+1)^2}{9(2\alpha+1)} - \mu\right)\left(\tfrac{1}{\alpha+1}\right)^2 \\ &\leq 9(\alpha+1)^2 - 9(2\alpha+1)\mu.\end{aligned}$$

Here, we use the result already proven for $\mu = \frac{4(\alpha+1)^2}{9(2\alpha+1)}$. Equality is attained on choosing $c_1 = c_2 = p_1 = p_2 = 2$ and $b_2 = b_3 = 1$ in (10).

Next, assume that $\frac{8(\alpha+1)^2}{9(2\alpha+1)} \leq \mu \leq \frac{4(\alpha+1)^2}{3(2\alpha+1)}$.

First, we deal with the case $\mu = \frac{4(\alpha+1)^2}{3(2\alpha+1)}$. It follows from (4),(5),(6) and (10) that

$$9(2\alpha+1)(\alpha+1)^2\left|a_3 - \tfrac{4(\alpha+1)^2}{3(2\alpha+1)}a_2^2\right| \leq 3(\alpha+1)^2 - \tfrac{(\alpha+1)^2}{4}(|c_1| - |p_1|)^2,$$

$$= \psi(|c_1|, |p_1|), \text{ say.}$$

A straightforward calculation shows that the ψ attains maximum value when $|c_1| = |p_1|$ and so

$$9(2\alpha+1)(\alpha+1)^2|a_3 - \mu a_2^2| \le 3(\alpha+1)^2.$$

Next, write

$$a_3 - \mu a_2^2 = \frac{9(2\alpha+1)\mu - 8(\alpha+1)^2}{4(\alpha+1)^2}\left(a_3 - \frac{4(\alpha+1)^2}{3(2\alpha+1)}a_2^2\right)$$

$$+ \frac{3(4(\alpha+1)^2 - 3(2\alpha+1)\mu)}{4(\alpha+1)^2}\left(a_3 - \frac{8(\alpha+1)^2}{9(2\alpha+1)}a_2^2\right)$$

and the result follows at once by using results already established for $\mu = \frac{8(\alpha+1)^2}{9(2\alpha+1)}$ and $\mu = \frac{4(\alpha+1)^2}{3(2\alpha+1)}$ above. The result is sharp for $p_2 = c_2 = 2, p_1 = c_1 = 0, b_2 = 0$ and $b_3 = \frac{1}{3}$ in (10).

Finally, consider $\mu \ge \frac{4(\alpha+1)^2}{3(2\alpha+1)}$.

Write

$$a_3 - \mu a_2^2 = a_3 - \frac{4(\alpha+1)^2}{3(2\alpha+1)}a_2^2 + \left(\frac{4(\alpha+1)^2}{3(2\alpha+1)} - \mu\right)a_2^2$$

and thus

$$9(2\alpha+1)(\alpha+1)^2|a_3 - \mu a_2^2| \le 9(2\alpha+1)(\alpha+1)^2\left|a_3 - \tfrac{4(\alpha+1)^2}{3(2\alpha+1)}a_2^2\right|$$

$$+ 9(2\alpha+1)(\alpha+1)^2\left(\mu - \tfrac{4(\alpha+1)^2}{3(2\alpha+1)}\right)|a_2|^2,$$

$$\le -9(\alpha+1)^2 + 9(2\alpha+1)\mu,$$

where results for $\mu = \frac{4(\alpha+1)^2}{3(2\alpha+1)}$ and the inequality $|a_2| \le \frac{1}{\alpha+1}$ have been used. By choosing $c_1 = p_1 = 2i, c_2 = p_2 = -2, b_2 = i$ and $b_3 = -1$ in (10), equality is obtained.

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