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Rational curves in Grassmannians and their Plücker embeddings

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Abstract. Fix integers $k > n \geq 2$ and $a_1 \geq \cdots \geq a_n$ such that $a_n \geq \lfloor (\binom{k}{n} - 1)/n \rfloor$ and $a_1 + \cdots + a_n + 1 \geq \binom{k}{n}$. Set $E := \oplus_{i=1}^{n} \mathcal{O}_{\mathbf{P}^1}(a_i)$. Let V be a general k-dimensional linear subspace of $H^0(\mathbf{P}^1, E)$. Here we prove that for all n-dimensional linear subspace W of V the evaluation map $W \otimes \mathcal{O}_{\mathbf{P}^1} \to E$ is an injection of sheaves. Equivalently, the natural map $\bigwedge^k(V) \to H^0(\mathbf{P}^1, \det(E))$ is injective.

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1. Introduction

Theorem 1. *Fix integers $k > n \geq 2$ and $a_1 \geq \cdots \geq a_n$ such that $a_n \geq \lfloor (\binom{k}{n} - 1)/n \rfloor$ and $a_1 + \cdots + a_n + 1 \geq \binom{k}{n}$. Set $E := \oplus_{i=1}^{n} \mathcal{O}_{\mathbf{P}^1}(a_i)$. Let V be a general k-dimensional linear subspace of $H^0(\mathbf{P}^1, E)$. Then for all n-dimensional linear subspace W of V the evaluation map $W \otimes \mathcal{O}_{\mathbf{P}^1} \to E$ is an injection of sheaves. Equivalently, the natural map $\bigwedge^k(V) \to H^0(\mathbf{P}^1, \det(E))$ is injective.*

Remark 1. The equivalence of the two statements appearing in Theorem 1 was pointed out by M. Teixidor i Bigas in [2]. Take E, V as in the statement of Theorem 1. Since E is spanned, $k > n$, and V is general, V spans E. Hence the pair (E, V) induces a morphism $h_{E,V} : \mathbf{P}^1 \to G(n, k)$, where $G(n, k)$ denote the Grassmannian of all $(k - n)$-dimensional linear subspaces of the vector space $\mathbb{K}^n$. Let $u_{n,k} : G(n, k) \to \mathbf{P}^{N(n,k)}$, $N(n, k) := \binom{n}{k} - 1$, be the Plücker embedding. In [2] M. Teixidor i Bigas also proved that Theorem 1 is equivalent

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to show that $u_{n,k} \circ h_{E,V}(\mathbf{P}^1)$ spans $\mathbf{P}^{N(n,k)}$. We will prove Theorem 1 proving the non-degeneracy of the curve $u_{n,k} \circ h_{E,V}(\mathbf{P}^1)$ inside $\mathbf{P}^{N(n,k)}$.

For a smooth curve of genus $g \ge 2$ a result similar to Theorem 1 was proved in [1] when E is a general degree n stable vector bundle with sufficiently high degree. We stress that in Theorem 1 we do not require that E is rigid, i.e. we do not require the inequality $a_1 \le a_n - 1$.

We work over an algebraically closed field $\mathbb{K}$.

2. Proof of Theorem 1

For all integers $b > a > 0$ let $G(a,b)$ denote the Grassmannian of all $((b-a)$-dimensional linear subspaces of $\mathbb{K}^{\oplus b}$. Thus $\dim(G(a,b)) = a(b-a)$ and there is a tautological exact sequence of vector bundles on $G(a,b)$

$$0 \to S_{G(a,b)} \to \mathcal{O}_{G(a,b)}^{\oplus b} \to Q_{G(a,b)} \to 0 \tag{1}$$

with $\operatorname{rank}(Q_{G(a,b)}) = b$, $\operatorname{rank}(S_{G(a,b)}) = b - a$ and $\det(Q_{G(a,b)}) \cong \det(S_{G(a,b)})^* \cong \mathcal{O}_{G(a,b)}(1)$, where $\mathcal{O}_{G(a,b)}(1)$ denotes the positive generator of $\operatorname{Pic}(G(a,b) \cong \mathbb{Z}$. $\mathcal{O}_{G(a,b)}(1)$ is very ample and the associated complete linear system $|\mathcal{O}_{G(a,b)}(1)|$ induces the Plücker embedding $u_{a,b}$ of $G(a,b)$ into $\mathbf{P}^{N(a,b)}$, $N(a,b) := \binom{b}{a} - 1$. We have $TG(a,b) \cong Q_{G(a,b)} \otimes S^*_{G(a,b)}$. For all subscheme Z of $G(a,b)$ let $N_{Z,G(a,b)}$ denote its normal sheaf in $G(a,b)$. Notice that $N_{Z,G(a,b)}$ is a spanned vector bundle if Z is smooth. We will often see $G(a,b)$ as the set of all $\mathbf{P}^{a-1}$'s contained in $\mathbf{P}^{b-1}$. $G(a,b)$ is a homogeous variety which contains many lines with respect to the Plücker embedding. Any such line is obtained in this way. Fix an a-dimensional linear subspace B of $\mathbf{P}^{b-1}$ and a codimension two linear subspace A of B (with the convenction $A = \emptyset$ if $a = 1$). Set $D(A,B) := \{D \in G(a,b) : A \subset D \subset B\}$. $D(A,B)$ is a line of $G(a,b)$ and the group $\operatorname{Aut}(G(a,b))$ acts transitively on the set $\Gamma(a,b)$ of all lines contained in $G(a,b)$. Take $D \in \Gamma$. The vector bundle $Q_{S(a,b)}|D$ is a direct sum of one line bundle of degree 1 and $a-1$ line bundles of degree 0. The vector bundle $S_{S(a,b)}|D$ is a direct sum of one line bundle of degree -1 and $b-a-1$ line bundles of degree 0. Thus $TG(a,b)|D$ is a direct sum of one line bundle of degree 2, $(a+b-2)$ line bundles of degree 1 and $(a-1)(b-a-1)$ line bundles of degree 0. Thus $N_{D,G(a,b)}$ is a direct sum of $(a+b-2)$ line bundles of degree 1 and $(a-1)(b-a-1)$ line bundles of degree 0.

The following lemma is well-known (see e.g. [1], §2).

Lemma 1. *Let $T \subset G(a,b)$ a reduced, connected and nodal curve such that $p_a(T) = 0$ and each irreducible component of T is a line. Then T is a smooth point of the Hilbert scheme $Hilb(G(a,b))$ of $G(a,b)$. Furthermore, T is smoothable, i.e. it is the flat limit of a family of smooth and connected rational curves contained in $G(a,b)$.*

Remark 2. Fix integers $b > a > 0$ and $d > 0$. Let $R(a,b,d)$ denote the set of all smooth and connected degree d rational curves contained in $G(a,b)$. For any $Z \in R(a,b,d)$ the vector bundle $N_{Z,G(a,b)}$ is a spanned vector bundle on $Z \cong \mathbf{P}^1$. Thus $h^1(Z, N_{Z,G(a,b)}) = 0$. Thus Z is a smooth point of the Hilbert scheme Hilb$(G(a,b))$. Furthermore, the vector bundles $Q_{G(a,b)}|Z$ and $S_{G(a,b)}|Z$ are rigid for any general element Z of any irreducible component of $R(a,b,d)$ (use the universal properties of the Grassmannians $G(a,b)$ and $G(b-a,a)$ and that every vector bundle on $\mathbf{P}^1$ is a flat limit of a family of rigid vector bundles).

Lemma 2. *Fix general P, Q in $G(a,b)$. There is a chain of $b-a$ lines joining Q with P, i.e. $b-a$ lines $D_1, \dots, D_{b-a}$ such that $P \in D_1$, $Q \in D_{b-a}$, $D_i \cap D_{i+1} \neq \emptyset$ for all $i \in \{1, \dots, b-a-1\}$, $T := D_1 \cup \cdots \cup D_{b-a}$ is connected and nodal and $p_a(T) = 0$.*

Proof. Since $G(a, a+1) \cong \mathbf{P}^a$, the case $b = a+1$ is trivial. Use induction on b and the description of all lines contained in $G(a,b)$. □

Lemma 3. *Fix integers $b > a > 0$. Then there exists a reduced, connected and nodal curve such that $\deg(T) = \binom{b}{a} - 1$, $p_a(T) = 0$, each irreducible component of T is a line and $u_{a,b}(T)$ spans $\mathbf{P}^{N(a,b)}$.*

Proof. Since $G(a, a+1) \cong \mathbf{P}^a$, the case $b = a+1$ is trivial. Use induction on b and Lemma 2. □

Proof of Theorem 1. Set $x := \lfloor (\binom{k}{n} - 1)/n \rfloor$ and $y := \binom{k,n}{-}1 - nx$. Let F be the rank n vector bundle on $\mathbf{P}^1$ isomorphic to the direct sum of y line bundles of degree $x+1$ and $n-y$ line bundles of degree x, i.e. the rigid line bundle with rank n and degree $\binom{k}{n}$. Notice that $h^0(\mathbf{P}^1, F) = \binom{k}{n} + n - 1 \geq k$. Our assumptions on a_n and $a_1 + \cdots + a_n$ imply the existence of an inclusion of sheaves $j : F \to E$. The map j induces an inclusion $j_* : H^0(\mathbf{P}^1, F) \to H^0(\mathbf{P}^1, E)$. By semicontinuity it is sufficient to prove the result for one k-dimensional linear subspace, e.g. one of the form $j_*(M)$ with M a general k-dimensional linear subspace of $H^0(\mathbf{P}^1, F)$. Apply Lemmas 1 and 3, Remark 2 and the interpretation of Theorem 1 given in the last part of Remark 1.

References

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