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Solution to Fredholm Fuzzy Integral Equations with Degenerate Kernel

M. M. Shamivand, A. Shahsavaran and S. M. Tari

Faculty of Science, Islamic Azad University
Borüjerd Branch, Borüjerd, Iran
m.shamivand@iaub.ac.ir

Abstract

The purpose of this paper is to present a new method to solve fuzzy Fredholm integral equations with degenerate kernel. Our results are given to demonstrate the proposed method and based on the concept concerning the crisp integral equations with degenerate kernel.

Keywords: Degenerate Kernel; Fredholm Fuzzy Integral Equation

1 Introduction

Fuzzy Fredholm integral equations have been solved with different methods [1-7]. The most remarkable properties of fuzzy set are that they are employed in many different research fields ranging from artificial intelligence and robotics, image processing, biological and medical science, applied operations research, economics and geogeraphy, quantum optics and gravity, socialogy, psychology and some more restricted topics.

2 Preliminaries

In this section the most used basic notations in fuzzy calcules are introduced.

Definition 1. A fuzzy number is a fuzzy set $u : \mathbb{R} \longrightarrow [0, 1]$ which satisfies:
1. u is upper semi-continuous.
2. $u(x) = 0$ outside some interval $[c, d]$.
3. there are real numbers a and b: $c \leq a \leq b \leq d$ for which,

3.1 $u(x)$ is monotonic increasing on $[c, a]$,
3.2 $u(x)$ is monotonic decreasing on $[b, d]$,

3.3 $u(x) = 1$, $a \leq x \leq b$.

An alternative parametric form of a fuzzy number which is equivalent to definition 1 is as follows: a fuzzy number u is completely determined by any pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r)$ and $\overline{u}(r)$, $0 \leq r \leq 1$, which satisfying the following three conditions:
i. $\underline{u}(r)$ is a bounded monotonic increasing left-continuous function.
ii. $\overline{u}(r)$ is a bounded monotonic decreasing left-continuous function.
iii. $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

For arbitrary $u = (\underline{u}, \overline{u})$ and $k \in \mathbb{R}$ we define addition and multiplication by k as:
addition:

$$(\underline{u+v})(r) = \underline{u}(r) + \underline{v}(r), (\overline{u+v})(r) = \overline{u}(r) + \overline{v}(r),$$

scalar multiplication:

$$(\underline{ku})(r) = \begin{cases} k\overline{u}(r), & k \geq 0 \\ k\underline{u}(r), & k < 0 \end{cases}$$

$$(\overline{ku})(r) = \begin{cases} k\underline{u}(r), & k \geq 0 \\ k\overline{u}(r), & k < 0 \end{cases}$$

The collection of all the fuzzy numbers with the addition and multiplication is denoted by E^1 and is a convex cone.

Definition 2. For arbitrary fuzzy numbers $u = (\underline{u}, \overline{u})$ and $v = (\underline{v}, \overline{v})$ the quantity

$$D(u, v) = \sup_{0 \leq r \leq 1} \{\max(|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|)\}$$

is the distance between u and v.

Definition 3. A function $f : \mathbb{R} \longrightarrow E^1$ is called a fuzzy function if for arbitrary fixed $t_0 \in \mathbb{R}$ and $\epsilon > 0$ there exist a $\delta > 0$ such that

$$|t - t_0| < \delta \Longrightarrow D(f(t), f(t_0)) < \epsilon,$$

f is said to be continuous.
We now define the integral of fuzzy function using the Riemann integral concept which is employed in the next section.

Definition 4. Let $f : [a, b] \longrightarrow E^1$ for each partition $P = \{t_0, t_1, ..., t_n\}$ of $[a, b]$ and for arbitrary $\xi_i, t_{i-1} \leq \xi_i \leq t_i, \quad 1 \leq i \leq n$, let

$$R_p = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}), \quad \lambda_p = max\{|t_i - t_{i-1}| : 1 \leq i \leq n\},$$

that $1 \leq p \leq n$, then definite integral of $f(t)$ over $[a,b]$ is

$$\int_a^b f(t)dt = \lim_{\lambda_p \to 0} R_p,$$

provided that this limit exists in the metric D.
If the fuzzy function $f(t)$ is continuous in the metric D, its definite integral exists. Furthermore,

$$\underline{(\int_a^b f(t,r)dt)} = \int_a^b \underline{f}(t,r)dt, \tag{1}$$

$$\overline{(\int_a^b f(t,r)dt)} = \int_a^b \overline{f}(t,r)dt. \tag{2}$$

Remark 1. Let $u(r) = (\underline{u}(r), \overline{u}(r)), \quad 0 \leq r \leq 1$, be a fuzzy number, we take

$$S_u(r) = \frac{\underline{u}(r) + \overline{u}(r)}{2}, \quad D_u(r) = \frac{\overline{u}(r) - \underline{u}(r)}{2}.$$

It is clear that $\underline{u}(r) = S_u(r) - D_u(r)$ and $\overline{u}(r) = S_u(r) + D_u(r)$.

Remark 2. Let $u(r) = (\underline{u}(r), \overline{u}(r))$ and $v(r) = (\underline{v}(r), \overline{v}(r)), \quad 0 \leq r \leq 1$, and also k and s are arbitrary real numbers. If $w = ku + sv$ then

$$S_w(r) = kS_u(r) + sS_v(r), \quad D_w(r) = |k|D_u(r) + |s|D_v(r).$$

Now by referring to remark 1, we have

$$|\overline{u}(r) - \overline{v}(r)| \leq |S_u(r) - S_v(r)| + |D_u(r) - D_v(r)|,$$

$$|\underline{u}(r) - \underline{v}(r)| \leq |S_u(r) - S_v(r)| + |D_u(r) - D_v(r)|.$$

Hence for all $r \in [a,b]$,

$$\max\{|\overline{u}(r) - \overline{v}(r)|, |\underline{u}(r) - \underline{v}(r)|\} \leq |S_u(r) - S_v(r)| + |D_u(r) - D_v(r)|,$$

so definition 2 yields

$$D(u,v) \leq \sup_{0 \leq r \leq 1} \{|S_u(r) - S_v(r)| + |D_u(r) - D_v(r)|\}.$$

Subsequently, $|S_u(r) - S_v(r)| \to 0$ and $|D_u(r) - D_v(r)| \to 0$ implies that $D(u,v) \to 0$.

3 Illustration of the method

In this section we present a new method for solving the linear fuzzy Fredholm integral equation with degenerate kernel. The proposed approach will be illustrated in terms of the following equation:

$$F(t) = f(t) + \lambda \int_a^b k(s,t)F(s)ds, \tag{3}$$

with $\lambda > 0$. It is assumed that kernel $k(s,t)$ is degenerate, that is,

$$k(s,t) = \sum_{i=1}^{n} a_i(s)b_i(t),$$

where $a_i(s)$ and $b_i(t)$, $\quad i = 1, 2, ..., n$, are linearly independent functions. In Eq. 3, if f is a crisp function then the solution is crisp as well, and in the case that f is a fuzzy function, we have Fredholm fuzzy integral equation of the second kind which may only process fuzzy solutions. Sufficient conditions for existence of a unique solution to the equation (1), where f is a fuzzy function, are given in [5].

Now we introduce parametric form of the fuzzy integral Eqs. (3) with respect to definition 2. Let $(\underline{f}(t,r), \overline{f}(t,r))$ and $(\underline{F}(t,r), \overline{F}(t,r))$ $(0 \le r \le 1, a \le t \le b)$ are parametric form of $f(t)$ and $F(t)$ respectively, then the parametric form of the fuzzy integral Eqs. (3) is as follows:

$$\underline{F}(t,r) = \underline{f}(t,r) + \lambda \int_a^b \underline{k(s,t)F(s,r)}ds, \tag{4}$$

$$\overline{F}(t,r) = \overline{f}(t,r) + \lambda \int_a^b \overline{k(s,t)F(s,r)}ds, \tag{5}$$

where,

$$\underline{k(s,t)F(s,r)} = \begin{cases} k(s,t)\underline{F}(s,r), & k(s,t) \ge 0 \\ k(s,t)\overline{F}(s,r), & k(s,t) < 0 \end{cases}$$

$$\overline{k(s,t)F(s,r)} = \begin{cases} k(s,t)\overline{F}(s,r), & k(s,t) \ge 0 \\ k(s,t)\underline{F}(s,r), & k(s,t) < 0 \end{cases}$$

By the above assumptions the Eqs. (4) and (5) will become in the following forms respectively.

$$\underline{F}(t,r) = \underline{f}(t,r) + \lambda \sum_{i=1}^{n} \int_a^b \underline{a_i(s)b_i(t)F(s,r)}ds, \tag{6}$$

$$\overline{F}(t,r) = \overline{f}(t,r) + \lambda \sum_{i=1}^{n} \int_a^b \overline{a_i(s)b_i(t)F(s,r)}ds. \tag{7}$$

If we denote by $A_i, \quad i = 1,2,...,n$ the set of union on subintervals of $[a,b]$ that $a_i(s)$ is nonnegative on these subintervals and by $B_i, \quad i = 1,2,...,n$ the set of union on subintervals of $[a,b]$ that $a_i(s)$ is negative on these subintervals. It is clear that $A_i \cup B_i = [a,b], \quad i = 1,2,...,n$.
Without loosing generality, we suppose that $b_i(t)$ is nonnegative for fixed t and $1 \leq i \leq n$. By the above assumptions we can rewrite Eqs. (6) and (7) respectively as follows:

$$\underline{F}(t,r) = \underline{f}(t,r) + \lambda \sum_{i=1}^{n} (\sum_{I_i \in A_i} \int_{I_i} a_i(s)b_i(t)\underline{F}(s,r)ds + \sum_{J_i \in B_i} \int_{J_i} a_i(s)b_i(t)\overline{F}(s,r)ds), \tag{8}$$

$$\overline{F}(t,r) = \overline{f}(t,r) + \lambda \sum_{i=1}^{n} (\sum_{I_i \in A_i} \int_{I_i} a_i(s)b_i(t)\overline{F}(s,r)ds + \sum_{J_i \in B_i} \int_{J_i} a_i(s)b_i(t)\underline{F}(s,r)ds). \tag{9}$$

By remark 1, remark 2 and above Eqs. we can affirm that

$$S_F(t,r) = S_f(t,r) + \lambda \sum_{i=1}^{n} \int_a^b a_i(s)b_i(t)S_F(s,r)ds, \tag{10}$$

$$D_F(t,r) = D_f(t,r) + \lambda \sum_{i=1}^{n} \int_a^b |a_i(s)||b_i(t)|D_F(s,r)ds. \tag{11}$$

Note that in the negative case of $b_i(t)$ for some i that same results as the above equations are obtained. It emerges that the technique of solving Eqs. (8) and (9) are dependent on the definitions of

$$c_i = \int_a^b a_i(s)S_F(s,r)ds,$$

$$d_i = \int_a^b |a_i(s)|D_F(s,r)ds.$$

By substituting c_i and d_i in (10) and (11) respectively, we obtain

$$S_F(t,r) = S_f(t,r) + \lambda \sum_{i=1}^{n} c_i b_i(t), \tag{12}$$

$$D_F(t,r) = D_f(t,r) + \lambda \sum_{i=1}^{n} d_i |b_i(t)|. \tag{13}$$

By substituting (12) into c_i we obtain

$$\begin{aligned}\sum_{i=1}^{n} c_i b_i(t) &= \sum_{i=1}^{n} b_i(t) \int_a^b a_i(s) S_F(s,r) ds \\ &= \sum_{i=1}^{n} b_i(t) \int_a^b a_i(s)(S_f(s,r) + \lambda \sum_{k=1}^{n} c_k b_k(s)) ds,\end{aligned}$$

therefore we get

$$\sum_{i=1}^{n} b_i(t) \{c_i - \int_a^b a_i(s)(S_f(s,r) + \lambda \sum_{k=1}^{n} c_k b_k(s)) ds\} = 0,$$

in a similar manner we get

$$\sum_{i=1}^{n} |b_i(t)| \{d_i - \int_a^b |a_i(s)|(D_f(s,r) + \lambda \sum_{k=1}^{n} d_k |b_k(s)|) ds\} = 0.$$

Since the functions $b_i(t)$ and consequently $|b_i(t)|, \quad i = 1, 2, ..., n$ are linearly independent, therefore,

$$c_i - \int_a^b a_i(s)(S_f(s,r) + \lambda \sum_{k=1}^{n} c_k b_k(s)) ds = 0, \tag{14}$$

$$d_i - \int_a^b |a_i(s)|(D_f(s,r) + \lambda \sum_{k=1}^{n} d_k |b_k(s)|) ds = 0. \tag{15}$$

For these computations in the sense of the unknowns, we simplify the problems by using the following notations

$$f_i^{(1)} = \int_a^b a_i(s) S_f(s,r) ds, \quad a_{ik}^{(1)} = \int_a^b a_i(s) b_k(s) ds,$$

$$f_i^{(2)} = \int_a^b |a_i(s)| D_f(s,r) ds, \quad a_{ik}^{(2)} = \int_a^b |a_i(s)||b_k(s)| ds,$$

where the constant $f_i^{(1)}, f_i^{(2)}, a_{ik}^{(1)}$ and $a_{ik}^{(2)} \quad (1 \leq i, k \leq n)$ are known then Eqs. (14) and (15) become respectively

$$c_i - \lambda \sum_{k=1}^{n} a_{ik}^{(1)} c_k = f_i^{(1)}, \quad i = 1, 2, ..., n, \tag{16}$$

$$d_i - \lambda \sum_{k=1}^{n} a_{ik}^{(2)} d_k = f_i^{(2)}, \quad i = 1, 2, ..., n, \tag{17}$$

which are two systems of n algebraic Eqs. for the unknowns c_i and d_i. Therefore the problem reduces to finding the quantities c_i and d_i, $\quad i = 1, 2, ..., n$. The determinants of these systems are two polynomials in term of λ of degree at most n. For all values of λ for which the determinants are not equal to zero the algebraic systems (16) and (17) have solution and thereby the fuzzy integral Eqs. (3) have a unique solutions. We can use Eqs. (12) and (13) and remark 1 to obtain the fuzzy solution of the problem.

4 Experimental results

This section illustrates how to implement the proposed method in order to obtain the solutions of the fuzzy Fredholm integral equations with degenerate kernel.

Example 1 [7]: Consider the fuzzy Fredholm integral equation with

$$\underline{f}(t, r) = \sin(\frac{t}{2})(\frac{13}{15}(r^2 - r) + \frac{2}{15}(4 - r^3 - r)),$$

$$\overline{f}(t, r) = \sin(\frac{t}{2})(\frac{2}{15}(r^2 + r) + \frac{13}{15}(4 - r^3 - r)),$$

$$k(s, t) = \frac{1}{10}\sin(s)\sin(\frac{t}{2}), \quad 0 \leq s, t \leq 2\pi,$$

also a=0, $b = 2\pi$ and λ=1. Using the assumptions described in the previous sections, we have

$$S_f(t, r) = \frac{1}{2}\sin(\frac{t}{2})(4 + r^2 - r^3)$$

$$D_f(t, r) = \frac{11}{30}\sin(\frac{t}{2})(4 - 2r - r^2 - r^3).$$

By considering $a_1(s) = \frac{1}{10}\sin(s), b_1(t) = \sin(\frac{t}{2})$ and using Eqs. (16) and (17), we have

$$c_1 = 0, d_1 = \frac{4}{30}(4 - 2r - r^2 - r^3).$$

From Eqs. (9) and (10),

$$S_f(t, r) = S_g(t, r) = \frac{1}{2}\sin(\frac{t}{2})(4 + r^2 - r^3),$$

$$D_f(t, r) = D_g(t, r) + d_1|b_1(t)| = \frac{1}{2}\sin(\frac{t}{2})(4 - 2r - r^2 - r^3)$$

hence by using remark 1, the solution is obtained

$$\underline{f}(t,r) = (4 - r - r^3)\sin(\frac{t}{2}),$$

$$\overline{f}(t,r) = (r + r^2)\sin(\frac{t}{2}),$$

which is the exact solution of the problem.

Example 2 [constructed]: Consider the fuzzy Fredholm integral equation with

$$\underline{f}(t,r) = rt - r - r^2, \quad \overline{f}(t,r) = r - rt - t^2,$$

$$k(s,t) = s + t, \quad 0 \leq s, t \leq 1,$$

and a=0, b=1 and λ=1. Using remark 1, we have

$$S_f(t,r) = -t^2, \quad D_f(t,r) = r - rt.$$

By considering $a_1(s) = s, a_2(s) = 1$, $b_1(t) = 1, b_2(t) = t$ and using Eqs. (16) and (17) we obtain

$$c_1 = \frac{17}{6}, \quad c_2 = 5,$$

$$d_1 = -3r, \quad d_2 = 5r,$$

from Eqs.(9) and (10) $S_f(t,r)$ and $D_f(t,r)$ are obtained

$$S_f(t,r) = -t^2 + 5t + \frac{17}{6},$$

$$D_f(t,r) = -2r - 6rt.$$

Hence by using remark 1, the solution is obtained

$$\underline{f}(t,r) = -t^2 + 5t + 6rt + 2r + \frac{17}{6},$$

$$\overline{f}(t,r) = -t^2 + 5t - 6rt - 2r + \frac{17}{6},$$

which is the exact solution of the problem.

References

[1] S. Abbasbandy, E. Babolian, and M. Alavi, Numerical method for solving linear Fredholm fuzzy integral equations of the second kind, Chaos, Solitons and Fractals, Vol. 31, pp. 138-146, 2007. .

[2] S. Abbasbandy, B. Asady, A not on "A new approach for difuzzifications, Fuzzy Sets and System, Vol. 128, pp. 131-132, 2002.

[3] M. Ma, A. Kandel and M. Fiedman, Correction to "A new approach for defuzzification, Fuzzy Sets and System, Vol. 128, pp. 133-134, 2002.

[4] W. Congsin, M. Ming, On embedding problem of fuzzy number spaces, fuzzy sets and systems ,44(1991), 33-38; 45(1992), 189-202; 46(1992),281-286. .

[5] W. Congsin, M. Ming, On the integrals, series and integrals equatins of fuzzy set-valued functions, J. of Harbin Inst. of Thechnology, 21(1990), 11-19.

[6] B. N. Datta, Numerical linear algebra and applications, Brooks/Cole Publishing Company, USA, 1995.

[7] M. Friedman, M. Ming, A. Kandel, Solution of fuzzy integral equations with arbitrary kernels, International Journal of Approximate Reasoning, 20(1999), 249-262.

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