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Characterization of Total Signed Graph and Semi-Total Signed Graphs[1]

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Abstract

A *signed graph* (or *sigraph* in short) is an ordered pair $S = (S^u, \sigma)$, where S^u is a graph $G = (V, E)$ and $\sigma : E \to \{+, -\}$ is a function from the edge set E of S^u into the set $\{+, -\}$. In this paper, we obtain characterization of semi-total line sigraph, semi-total point sigraph and total sigraph of a sigraph S.

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1 Introduction

For standard terminology and notation in graph theory we refer Harary [6] and West [10] and Zaslavsky [11, 12] for sigraphs. Throughout the text, we consider finite, undirected graph with no loops or multiple edges. A *signed graph* (or *sigraph* in short; see [5]) is an ordered pair $S = (S^u, \sigma)$, where S^u is a graph $G = (V, E)$, called the *underlying graph* of S and $\sigma : E \to \{+, -\}$ is a function from the edge set E of S^u into the set $\{+, -\}$, called the *signature* of S. Alternatively, the sigraph can be written as $S = (V, E, \sigma)$, with V, E, σ in the above sense. Let $E^+(S) = \{e \in E(G) : \sigma(e) = +\}$ and $E^-(S) = \{e \in E(G) : \sigma(e) = -\}$. The elements of $E^+(S)$ and $E^-(S)$ are called *positive* and *negative* edges of S, respectively. A sigraph is said to be *homogeneous* if all its edges are of the same sign and *heterogeneous* otherwise.

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For a sigraph S, Gill [4] defined its $\times$-*line sigraph* $L_\times(S)$ as follows: the $L_\times(S)$ is a sigraph defined on the line graph $L(S^u)$ of the graph S^u by assigning to each edge ef of $L(S^u)$, the product of signs of the adjacent edges e and f of S. A cycle in a sigraph S is said to be *positive* if it contains an even number of negative edges. A given sigraph S is said be *balanced* if every cycle in S is positive [5]. The sign of a cycle in S is the product of sign of all the edges contained in the cycle. Thus, alternatively a cycle in S is positive if its sign is positive. The following important lemma on balanced sigraph is given by Zaslavsky:

Lemma 1.1 [13] *A signed graph in which every chordless cycle is positive, is balanced.*

The *total graph* $T(G)$ [2, 3] of a graph G is that graph whose vertex set is $V(G) \cup E(G)$ where $V(G)$ and $E(G)$ are vertex set and edge set of G, respectively and in $T(G)$ two vertices are adjacent if and only if they are adjacent or incident in G. The *semi-total line graph* $T_1(G)$ [7] of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ where $V(G)$ and $E(G)$ are vertex set and edge set of G, respectively and in $T_1(G)$ two vertices are adjacent if and only if (i) they are adjacent edges in G (ii) one is a vertex and the other is an edge in G incident to it. The *semi-total point graph* $T_2(G)$ [7] of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ where $V(G)$ and $E(G)$ are vertex set and edge set of G, respectively and in $T_2(G)$ two vertices are adjacent if and only if (i) they are adjacent vertices in G, (ii) one is a vertex and the other is an edge in G incident to it.

Let $S = (V,\ E,\ \sigma)$ be any sigraph. Its *total sigraph* $T(S)$ [as shown in Figure 1] has $T(S^u)$ as its underlying graph and for any edge uv of $T(S^u)$

$$\sigma_T(uv) = \begin{cases} \sigma(uv) & \text{if } u, v \in V \\ \sigma(u)\sigma(v) & \text{if } u, v \in E \\ \sigma(u) \prod\limits_{e_j \in E_v} \sigma(e_j) & \text{if } u \in E \textit{ and } v \in V. \end{cases}$$

Let $S = (V,\ E,\ \sigma)$ be any sigraph. Its *semi-total line sigraph* $T_1(S)$ [as shown in Figure 1] has $T_1(S^u)$ as its underlying graph and for any edge uv of $T_1(S^u)$

$$\sigma_{T_1}(uv) = \begin{cases} \sigma(u)\sigma(v) & \text{if } u, v \in E \\ \sigma(u) & \text{if } u \in E \textit{ and } v \in V. \end{cases}$$

Let $S = (V,\ E,\ \sigma)$ be any sigraph. Its *semi-total point sigraph* $T_2(S)$ [as shown in Figure 1] has $T_2(S^u)$ as its underlying graph and for any edge uv of $T_2(S^u)$

$$\sigma_{T_2}(uv) = \begin{cases} \sigma(uv) & \text{if } u, v \in V \\ \sigma(u) \prod\limits_{e_j \in E_v} \sigma(e_j) & \text{if } u \in E \textit{ and } v \in V. \end{cases}$$

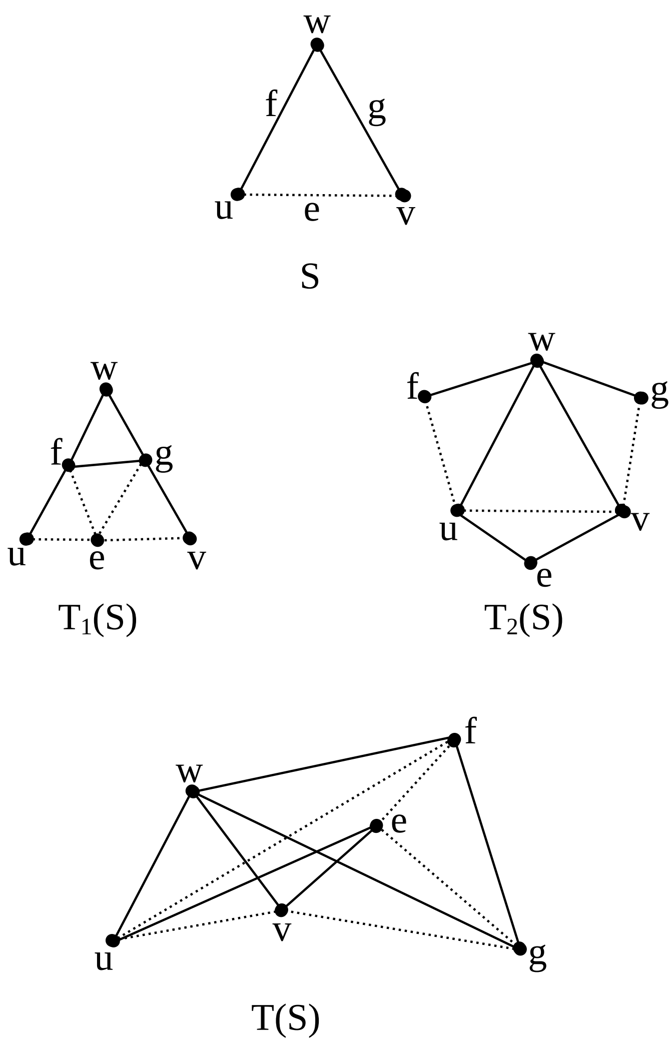

Figure 1: Showing $T_1(S)$, $T_2(S)$ and $T(S)$ of S

We observe that $L_\times(S)$ is an induced subsigraph of $T_1(S)$ and S is an induced subsigraph of $T_2(S)$. Also, S and $L_\times(S)$ are the induced subsigraphs of $T(S)$. Now the characterization of certain sigraph equations has been studied in [9]. Here we study the characterization of some more sigraph equations.

2 Characterization of semi-total line sigraph

Let u be an arbitrary vertex of a graph G. We denote the set consisting of u and all vertices of G adjacent with u by $\overline{N}(u)$. This set is called the *closed neighborhood* of u. If G and H are graphs with the property that the identification of any vertex of G with an arbitrary vertex of H results in a unique graph(upto isomorphism), then we write $G.H$ for this graph [6]. Sampathkumar obtained the following characterization of semi-total line graph:

Theorem 2.1 [7] *A connected graph* G *is a semi-total line graph of a graph* H*, i.e.* $G = T_1(H)$ *for some graph* H *if and only if* $V(G)$ *can be partitioned into two subsets* S_1 *and* S_2 *such that for every* $w \in S_1$ *there exists a unique pair of distinct vertices* $u, v \in S_2$ *satisfying the conditions:* (i) u *and* v *are adjacent to* w *and deg* $w =$ *deg* $u +$ *deg* v*, and* (ii) *the subgraph of* G *induced by* $\overline{N}(u) \cup \overline{N}(v)$ *is* $K_{m+1}.K_{n+1}$*, where deg* $u = m$ *and deg* $v = n$*.*

Theorem 2.2 [1] *The* $\times$*-line sigraph* $L_\times(S)$ *of a sigraph* S *is balanced.*

Now, we give the characterization of semi-total line sigraph of a sigraph.

Theorem 2.3 *A connected sigraph* $S_1 = (S_1^u, \sigma_1)$ *is a semi-total line sigraph of a sigraph if and only if the following conditions hold in* S_1*:*

(i) S_1^u *is a semi-total line graph,*

(ii) S_1 *is balanced, and*

(iii) *the vertex set* $V(S_1)$ *of* S_1 *can be partitioned into two subsets* V_1 *and* V_2 *such that for every* $w \in V_1$ *there exists a unique pair of distinct vertices* $u, v \in V_2$ *satisfying that* $\sigma_1(uw) = \sigma_1(vw)$*.*

Proof: **Necessity**: Suppose a connected sigraph $S_1 = (S_1^u, \sigma_1)$ is a semi-total line sigraph of a sigraph $S_2 = (S_2^u, \sigma_2)$ i.e. $S_1 = T_1(S_2)$. Since $S_1 = T_1(S_2)$, therefore $S_1^u = T_1(S_2^u)$. Thus, (i) follows. Next, by the definition of $T_1(S_2)$, S_1 contains $L_\times(S_2)$ as an induced subsigraph, triangles due to the adjacent edges e_i and e_j in S_2 and the vertex v such that $e_i \cap e_j = \{v\}$ and cycles formed by the symmetric difference of these triangles and cycles in $L_\times(S)$. Since $L_\times(S_2)$ is a balanced sigraph due to Theorem 2.2, it remains to show that triangle T with edges $e_ie_j,\ e_iv$ *and* e_jv is balanced. Now, the sign of triangle T is

$$= \sigma_1(e_ie_j)\sigma_1(e_iv)\sigma_1(e_jv)$$
$$= \sigma_1(e_i)\sigma_1(e_j)\sigma_1(e_i)\sigma_1(e_j)$$
$$= (\sigma_1(e_i))^2(\sigma_1(e_j))^2 = +.$$

Thus, T is balanced. So due to Lemma 1.1, S_1 is balanced and hence, (ii) follows. Now, in the process of construction of S_1 from S_2, let w be the new vertex introduced on an edge uv of S_2, then by the definition of $T_1(S_2)$,

$\sigma_1(uw) = \sigma_1(vw)$ and therefore, (iii) follows.

Sufficiency: Suppose conditions (i), (ii) and (iii) hold for a given sigraph S_1. Now we construct a sigraph S_2 on V_2 as follows: By the condition (i), $S_1^u = T_1(S_2^u)$, therefore two vertices $u, v \in V_2$ are adjacent if and only if there exists a vertex $w \in V_1$ such that $d(w) = d(u) + d(v)$ and $\sigma_2(uv) = \sigma_1(uw)$ *or* $\sigma_1(vw)$ being $\sigma_1(uw) = \sigma_1(vw)$. Thus, we can verify that $T_1(S_2) = S_1$. Hence the theorem.

3 Characterization of semi-total point sigraph

Sampathkumar obtained the following characterization of semi-total point graph:

Theorem 3.1 [8] *Let G be a graph with $3k$; $k \geq 1$ lines, then G is a semi-total point graph of a graph H, i.e. $G = T_2(H)$ for some graph H if and only if (i) every line of G lies on a triangle, and (ii) if the triangle T in G has no point of degree two, then for every line x of T, there should be exactly one triangle having a point of degree two and containing x.*

Here, we obtain a characterization of semi-total point sigraph.

Theorem 3.2 *A connected sigraph $S_1 = (S_1^u, \sigma_1)$ is a semi-total point sigraph of a sigraph if and only if the following conditions hold in S_1:*

(i) *S_1^u semi-total point graph,*

(ii) *the vertex set $V(S_1)$ of S_1 can be partitioned into two subsets V_1 and V_2 such that for every $w \in V_1$ of degree two there exists a unique pair of distinct vertices $u, v \in V_2$ satisfying:*

(a) $\sigma_1(uw) = \sigma_1(uv) \prod_{e_j \in E_u} \sigma_1(e_j)$, *and*

(b) $\sigma_1(vw) = \sigma_1(uv) \prod_{e_j \in E_v} \sigma_1(e_j)$.

Proof: **Necessity**: Suppose a connected sigraph $S_1 = (S_1^u, \sigma_1)$, is a semi-total point sigraph of a sigraph $S_2 = (S_2^u, \sigma_2)$ i.e. $S_1 = T_2(S_2)$. Since $S_1 = T_2(S_2)$, therefore $S_1^u = T_2(S_2^u)$. Thus, (i) follows. Now, in the process of construction of S_1 from S_2, we insert a triangle on each edge $e_j = uv$ of S_2 with a new vertex $w(e_j)$ (or w) as its vertex. Clearly, the newly introduced vertex w is different for different edges giving $d(w) = 2$. Now V_1 is the set containing such type of vertices w and V_2 contains remaining vertices of S_1. Now by the definition of $T_2(S_2)$, $\sigma_1(uw) = \sigma_1(uv) \prod_{e_j \in E_u} \sigma_1(e_j)$ and $\sigma_1(vw) = \sigma_1(uv) \prod_{e_j \in E_v} \sigma_1(e_j)$. Thus, $(ii)(a)$ and $(ii)(b)$ follows.

Sufficiency: Suppose conditions (i) and (ii) hold for a given sigraph S_1. Now we construct a sigraph S_2 such that $T_2(S_2) = S_1$ as follows: Let H be the set of all independent vertices of V_1. Now we obtain S_2 from S_1 by removing the set of vertices H. Thus, we easily verify that $T_2(S_2) = S_1$. Hence the theorem.

4 Characterization of total sigraph

Behzad and Radjavi obtained the following characterization of regular total graph:

Theorem 4.1 [3] *A connected regular graph G, $G \neq T(C)$, $T(K)$, is a total graph if and only if (i) it has order $p(1 + d/2)$ and degree $2d$ for some positive integers p and d, where $2 < d < p - 1$, (ii) it has exactly p special points, and (iii) $G = G_o$, where G_o is the subgraph of G generated by its special points.*

Theorem 4.2 [3] *Let H be a complete graph of order p, $p > 3$, and let $G = T(H)$. Then G has exactly $p + l$ subgraphs G_i, $i = 1, 2, ..., p + l$, isomorphic with H and $G = T(G_i)$ for each i.*

Behzad obtained the following characterization of total graph:

Theorem 4.3 [2] *Assume that H, $H \neq T(C)$, $T(K)$, is a connected graph, and that u is an arbitrary vertex of H. Then H is total if and only if $H = T(G_{v;u_l v_1})$ for some $v \in \overline{N}(u)$ and some edge $u_1 v_1$, where u_l and v_1 are two even vertices of H adjacent with v.*

Now, we obtain a characterization of total sigraph.

Theorem 4.4 *A connected sigraph $S_1 = (S_1^u, \sigma_1)$ is a total sigraph of a sigraph if and only if the following conditions hold in S_1:*

- (i) *S_1^u is a total graph,*
- (ii) *the vertex set $V(S_1)$ of S_1 can be partitioned into two subsets V_1 and V_2 such that for every $w \in V_1$ there exists a unique pair of distinct adjacent vertices u, $v \in V_2$ satisfying:*
  - (a) $\sigma_1(uw) = \sigma_1(uv) \prod_{e_j \in E_u} \sigma_1(e_j)$,

(*b*) $\sigma_1(vw) = \sigma_1(uv) \prod_{e_j \in E_v} \sigma_1(e_j)$, *and*

(*iii*) *subsigraph induced by vertices of* V_1 *is* $\times$*-line sigraph of subsigraph induced by vertices of* V_2.

Proof: **Necessity**: Suppose a connected sigraph $S_1 = (S_1^u, \sigma_1)$ is a total sigraph of a sigraph $S_2 = (S_2^u, \sigma_2)$ i.e. $S_1 = T(S_2)$. Since $S_1 = T(S_2)$, therefore $S_1^u = T(S_2^u)$. Thus, (*i*) follows. Now, in the process of construction of S_1 from S_2, we insert a triangle on each edge $e_j = uv$ of S_2 with a new vertex $w(e_j)$ (or w) as its vertex. Clearly, the newly introduced vertex w is different for different edges. Now V_1 is the set of vertices containing such w's and V_2 contains remaining vertices of S_1. Now by the definition of $T(S_2)$, $\sigma_1(uw) = \sigma_1(uv) \prod_{e_j \in E_u} \sigma_1(e_j)$ and $\sigma_1(vw) = \sigma_1(uv) \prod_{e_j \in E_v} \sigma_1(e_j)$. Thus, (*ii*)(*a*) and (*ii*)(*b*) follows. Again, by the definition of $T(S_2)$, subsigraph induced by vertices of V_1 is $\times$-line sigraph of subsigraph induced by vertices of V_2 and therefore, (*iii*) follows.

Sufficiency: Suppose conditions (*i*), (*ii*) and (*iii*) hold for a given sigraph S_1. Now we construct a sigraph S_2 such that $T(S_2) = S_1$ as follows: Let H be the subgraph induced by the vertices of V_1. Now we obtain S_2 from S_1 by removing all the vertices of H. Thus, we can verify that $T_2(S_2) = S_1$. Hence the theorem.

5 Conclusion and scope

In this paper, we have obtained the characterization of semi-total line sigraph, semi-total point sigraph and total sigraph. Balance and consistency of these structures with respect to particular marking have been studied else where in different research papers and one can find more properties of these structures once the characterizations are available.

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