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A Condition for the Equivalence of I- and I^*-Convergence in 2-Normed Spaces

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Abstract

In this article we define and investigate the concepts of I and I^* -convergence for double sequences in 2-normed space and derive several basic properties of these concepts in 2-normed space. Finally,we consider the condition (AP2) of I and I-convergence of double sequences in 2-normed spaces and prove some basic properties.

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1 Introduction

The concept of statistical convergence was introduced over nearly the last fifty years(Fast (1951), Schoenberg(1959)),it has become an active area of research in recent years .The idea of I-convergence was informally introduced by Kostyrko et al (2001) and also independently by Nuray and Ruckle (2000) , as a generalization of statistical convergence.Ideal convergence provides a general framework to study the properties of various types of convergence.
Very recently some works on I-convergence of double sequences have also been done (see [5,6,21,26]). The notion of linear 2-normed spaces has been investigated by Gâhler in1960's [11,12] and has been developed extensively in different subjects by others[3,14,23,27].It seems therefore reasonable to investigate the concepts of $I$and I^*-convergence for the double sequences in 2-normed spaces.
Throughout this paper $\mathbb{N}$ will denote the set of positive integers. Let $(X, \|.\|)$ be a normed space. Let E be subset of positive integers $\mathbb{N}$ and $j \in \mathbb{N}$.The quotient $d_j(E) = card(E \bigcap \{1, ..., j\})/j$ is called the j'th *partial density* of K.Note that d_j is a probability measure on $\mathcal{P}(\mathbb{N})$, with support $\{1, ..., j\}$[2,4,7,26].
The limit d(E)=$\lim_{j\to\infty} d_j(K)$ is called the *natural density* of $E \subseteq \mathbb{N}$(if exists). Clearly, finite subsets have natural density zero and $d(E^c) = 1 - d(E)$ where $E^c = E - \mathbb{N}$, i.e., the complement of E [2,24].
Recall that a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X is said to be statistically convergent to $l \in X$if the set$A(\epsilon)$=$\{n \in \mathbb{N} : \|x_n - l\| \geq \epsilon\}$has natural density zero for each $\epsilon > 0$ in other words for each $\varepsilon > 0$,

$$\lim_n \frac{1}{n} card(\{k \leq n : |x_k - l| \geq \varepsilon\}) = 0$$

and x=$(x_n)_{n\in\mathbb{N}}$ is called to be statistically Cauchy sequence if for each $\varepsilon > 0$ there exists a number $N = N(\varepsilon)$ such that

$$\lim_n \frac{1}{n} card(\{k \leq n : |x_k - x_N| \geq \varepsilon\}) = 0$$

The convergence of a double sequence introduce by many manner[4,21,22]. By the convergence of a double sequence we mean the convergence in Pringsheim's sense [22]. A double sequence $x{=}(x_{jk})_{j,k\in\mathbb{N}}$ is called to be *convergent in the Pringsheim's sense* if for each $\varepsilon > 0$ there exist a positive integer $N = N(\varepsilon)$ such that for all $j, k \geq N$ implies $|x_{jk} - l| < \varepsilon$. L is called the Pringsheim limit of x.

Let $A \subseteq \mathbb{N} \times \mathbb{N}$ be a set of positive integers and let $A(n, m)$ be the numbers of (j, k) in A such that $j \leq n$ and $k \leq m$. Then the two-dimensional concept of *natural density* can be defined as follows.
The *lower asymptotic density* of a set $A \subseteq \mathbb{N} \times \mathbb{N}$ is defined as

$$\underline{d_2}(A) = \liminf_{n,m} \frac{A(n,m)}{nm}$$

If the sequence $(\frac{A(n,m)}{nm})_{n,m\in\mathbb{N}}$ has a limit in Pringsheim's sense then we say that A has a *double natural density* and is defined as

$$d_2(A) = \lim_{n,m} \frac{A(n,m)}{nm}$$

Next we recall the following definition ,where Y represents an arbitrary set.

Definition 1.1. A family $\mathcal{I} \subseteq \mathcal{P}(Y)$ of subsets a nonempty set Y is said to be an ideal in Y if:
i) $\emptyset \in \mathcal{I}$
ii) $A, B \in \mathcal{I}$ implies $A \bigcup B \in \mathcal{I}$
iii) $A \in \mathcal{I}, B \subseteq A$ implies $B \in \mathcal{I}$
$\mathcal{I}$ is called a nontrivial ideal if $X \notin \mathcal{I}$.

Definition 1.2. Let $Y \neq \emptyset$.Anon empty family F of subsets of Y is said to be a *filter* in Y provided:
i) $\emptyset \in F$.
ii) $A, B \in F$ implies $A \cap B \in F$.
iii)$A \in F, A \subseteq B$ implies $B \in F$.
If $\mathcal{I}$ is a nontrivial ideal in Y,$Y \neq \emptyset$,then the class

$$F(\mathcal{I}) = \{M \subset Y : (\exists A \in \mathcal{I})(M = Y - A)\}$$

is a filter on Y ,called the *filter associated* with $\mathcal{I}$.

Definition 1.3. A nontrivial ideal $\mathcal{I}$ in Y is called *admissible* if $\{x\} \in \mathcal{I}$ for each $x \in Y$.

Definition 1.4. A nontrivial ideal $\mathcal{I}$ in $\mathbb{N}\times\mathbb{N}$ is called *strongly admissible* if $\{i\}\times\mathbb{N}$ and $\mathbb{N}\times\{i\}$ belong to $\mathcal{I}$ for each $i\in\mathbb{N}$.
It is evident that a strongly admissible ideal is admissible also.
Let $\mathcal{I}_0$=$\{A\subset\mathbb{N}\times\mathbb{N}:(\exists m(A)\in\mathbb{N})(i,j\geq m(A)\Rightarrow(i,j)\in\mathbb{N}\times\mathbb{N}-A)\}$.
Then $\mathcal{I}_0$ is a nontrivial strongly admissible ideal and clearly an ideal $\mathcal{I}$ is strongly admissible if and only if $\mathcal{I}_0\subseteq\mathcal{I}$.[5]

Let $\mathcal{I}\subseteq\mathcal{P}(\mathbb{N})$ be a nontrivial ideal in $\mathbb{N}$.The sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be $\mathcal{I}$-convergent to $x\in X$, if
for each $\epsilon>0$ the set $A(\epsilon)$=$\{n\in\mathbb{N}:\|x_n-x\|\geq\epsilon\}$ belongs to $\mathcal{I}$ [1,17,19].

2 Preliminary Notes

In [10], the concept of I^*-convergence was also introduced and a detailed study was made to explore its relation with I-convergence. We introduce the concepts $\mathcal{I}$ and $\mathcal{I}^*$-convergence of double sequences in metric space X is follow.

Definition 2.1. A double sequence x=$(x_{jk})_{j,k\in\mathbb{N}}$ of elements of X is said to be $\mathcal{I}$-convergent to $l\in X$ if for every $\varepsilon>0$ we have $A(\varepsilon)\in I$,
where $A(\varepsilon)$=$\{(m,n)\in\mathbb{N}\times\mathbb{N}:\rho(x_{mn},l)\geq\varepsilon\}$and we write it as

$$\mathcal{I}-\lim_{m,n}x_{mn}=l$$

Definition 2.2. A double sequence x=$(x_{jk})_{j,k\in\mathbb{N}}$ of elements of X is said to be $\mathcal{I}^*$-convergent to $l\in X$ if there exists a set $M\in F(\mathcal{I})$ (i.e.$\mathbb{N}\times\mathbb{N}-M\in\mathcal{I}$) such that

$$\lim_{m,n}x_{mn}=l \qquad (m,n)\in M$$

We write it as

$$\mathcal{I}^*-\lim_{m,n}x_{mn}=l$$

The notion of linear 2-normed spaces has been investigated by Gâhler in1960's [11,12] and has been developed extensively in different subjects by others[3,14,23]. Let X be a real linear space of dimension greater than 1,and $\|.,.\|$ be a non-negative real-valued function on $X\times X$ satisfying the following conditions:

G1)$\|x, y\| = 0$ if and only if x and y are linearly dependent vectors.

G2) $\|x, y\| = \|y, x\|$ for all x,y in X.

G3)$\|\alpha x, y\| = |\alpha|\|x, y\|$ where α is real

G4)$\|x + y, z\| \leq \|x, z\| + \|y, z\|$ for all x, y, z in X

$\|., .\|$ is called a 2-norm on X and the pair $(X, \|., .\|)$ is called a linear 2-normed space. In addition, for all scalars α and all x, y, z in X, we have the following properties :

1)$\|., .\|$ is nonnegative.

2)$\|x, y\| = \|x, y + \alpha x\|$

3)$\|x - y, y - z\| = \|x - y, x - z\|$

Some of the basic properties of 2-norm introduce in [24].
As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x, y\| :=$ the area of the parallelogram spanned by the vectors x and y, which may be given clearly by the formula

$$\|x, y\| = |x_1 y_2 - x_2 y_1| \quad , \quad x = (x_1, x_2) \quad y = (y_1, y_2) \tag{2.1}$$

Given a 2-normed space $(X, \|., .\|)$, one can derive a topology for it via the following definition of the limit of a sequence: a sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be convergent to x in X if $\lim_{n\to\infty} \|x_n - x, z\| = 0$ for every $z \in X$ .This can be written by the formula:

$$(\forall z \in Y)(\forall \epsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \geq n_0) \qquad \|x_n - x, z\| < \epsilon$$

We write it as

$$x_n \xrightarrow{\|.,.\|_X} x$$

Definition 2.3. A sequence $(x_n)n \in \mathbb{N}$ is a Cauchy sequence in a 2-normed space $(X, \|., .\|)$ if $\lim_{n,m} \|x_n - x_m, z\| = 0$ for every $z \in X$.
Recall that $(X, \|., .\|)$ is a 2-Banach space, if every Cauchy sequence in X is convergence to some $x \in X$.

Definition 2.4. Let $(X, \|., .\|)$ be 2-normed space and $x \in X$.We say that x is an *accumulation point* of X if there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of distinct elements of X such that $x_k \neq x$ (for any k) and $x_n \xrightarrow{\|.,.\|_X} x$

Lemma 2.5. *[15]Let $v = \{v_1, ..., v_k\}$ be a basis of X. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is convergent to x in X if and only if $\lim_{n\to\infty} \|x_n - x, v_i\| = 0$ for every $i = 1, ..., k$. We can define the norm $\|.\|_\infty$ on X by*

$$\|x\|_\infty := \max\{\|x, v_i\| : i = 1, ..., d = k\}$$

Associated to the derived norm $\|.\|_\infty$, we can define the(open) balls $B_{v_1,v_2,...,v_n}(\boldsymbol{x}, r)$=$B_v(\boldsymbol{x}, r)$ centered at $\boldsymbol{x}$ having radius r by

$$B_v(\boldsymbol{x}, r) := \{y : \|\boldsymbol{x} - y\|_\infty < r\}$$

Lemma 2.6. *[15]A sequence $(x_n)_{n\in\mathbb{N}}$ in X is convergent to x in X if and only if*

$$\lim_{n\to\infty} \|x_n - x\|_\infty = 0$$

Example 2.7. Let $X = \mathbb{R}^2$ be equipped with the 2-norm $\|x, y\|$:= the area of the parallelogram spanned by the vectors x and y, which may be given explicitly by the formula

$$\|x, y\| = |x_1y_2 - x_2y_1| \qquad , \qquad x = (x_1, x_2) \qquad y = (y_1, y_2)$$

Take the standard basis $\{i, j\}$ for $\mathbb{R}^2$.

Then, $\|x, i\| = |x_2|$ and $\|x, j\| = |x_1|$, and so the derived norm $\|.\|_\infty$ with respect to $\{i, j\}$ is

$$\|x\|_\infty = \max\{|x_1|, |x_1|\}, \qquad x = (x_1, x_2)$$

Thus, here the derived norm $\|.\|_\infty$ is exactly the same as the uniform norm on $\mathbb{R}^2$.Since the derived norm is norm,it is equivalent to Euclidean norm on $\mathbb{R}^2$.

3 Main Results

Now ,we introduce the I_2 and I_2^*-convergence for double sequences in 2-normed spaces and so we extend this concepts to I_2-limit points and I_2-cluster points in this spaces.We inspire the follow definition of
Pringsheim 痴 sense[22].

Definition 3.1. Let $x{=}(x_{jk})_{j,k\in\mathbb{N}}$ be a double sequence in 2-normed space $(X, \|.,.\|)$.A double sequence $x{=}(x_{jk})_{j,k\in\mathbb{N}}$ is said to be *convergent to* $l \in X$ if $(\forall z \in X)(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall j,k \geq N) \qquad \|x_{jk} - l, z\| < \varepsilon$
We write it as

$$x_{jk} \xrightarrow{\|.,.\|_X} l$$

A double sequence $x{=}(x_{jk})_{j,k\in\mathbb{N}}$ is said to be bounded if for each nonzero $z \in X$ and for all $j,k \in \mathbb{N}$ there exists $M > 0$ such that $\|x_{jk}, z\| < M$.
Note that a convergent double sequence need not be bounded.

Now we define the I_2 and I_2^*-convergence for double sequence $x{=}(x_{jk})_{j,k\in\mathbb{N}}$ as follows:

Definition 3.2. A double sequence $x{=}(x_{jk})_{j,k\in\mathbb{N}}$ in 2-normed space $(X, \|.,.\|)$ is said to be I_2-convergence to $l \in X$,if for all $\varepsilon > 0$ and nonzero $z \in X$,the set

$$A(\varepsilon) = \{(j,k) : \|x_{jk} - l, z\| \geq \varepsilon\} \in I_2$$

In this case we write it as

$$I_2 - \lim_{j,k} x_{jk} = l$$

Remark 3.3. Put $I_d{=}\ \{A \subset \mathbb{N} \times \mathbb{N} : d_2(A) = 0\}$. Then I_d is an admissible ideal in $\mathbb{N} \times \mathbb{N}$ and I_{d2}-convergence becomes statistical convergence[24].

Remark 3.4. If $x{=}(x_{jk})_{j,k\in\mathbb{N}}$ is I_2-convergent,then $(x_{jk})_{j,k\in\mathbb{N}}$need not be convergent .Also it is not necessarily bounded.This actuality can be seen from the next example.

Definition 3.5. A double sequence $x{=}(x_{jk})_{j,k\in\mathbb{N}}$ in 2-normed space $(X, \|.,.\|)$ is said to be I_2^*-convergence to $l \in X$,if there exists a set $M \in F(I)$ (i.e.$N \times N \backslash M \in I$) such that $\lim_{m,n} x_{mn}{=}l$ $(m,n) \in M$ and we write it

$$I_2^* - \lim_{j,k} x_{jk} = l$$

Example 3.6. Let $(X, \|.,.\|)$ be 2-normed space introduced in Example 2.3 and the $x{=}(x_{jk})_{j,k\in\mathbb{N}}$ be defined as

(1,1) otherwise

and let $\qquad l = (1, 1).$

Then for every $\varepsilon > 0$ and $z \in X$
$\{(j,k), j \leq n, k \leq m : \|x_{jk} - l, z\| \geq \varepsilon\} \subseteq \{1, 4, 9, 16, ..., j^2, ...\} \times \{1, 4, 9, 16, ..., k^2, ...\}$.
Hence
the cardinality of the set $\{(j,k), j \leq n, k \leq m : \|x_{jk} - l, z\| \geq \varepsilon\} \leq \sqrt{j}\sqrt{k}$ for each $\varepsilon > 0$.
This implies $d_2(\{(j,k), j \leq n, k \leq m : \|x_{jk} - l, z\| \geq \varepsilon\}) = 0$ for each $\varepsilon > 0$ and $z \in X$.We have

$$I_{d2} - \lim_{j,k} x_{jk} = l$$

.
But x=$(x_{jk})_{j,k\in\mathbb{N}}$ is neither convergent to l nor bounded.

Remark 3.7. The following corollary can be verifies that if x=$(x_{jk})_{j,k\in\mathbb{N}}$ be I_2-convergent to $l \in X$, then l is determined uniquely.

Corollary 3.8. *let x=$(x_{jk})_{j,k\in\mathbb{N}}$ is a convergent double sequence in 2-normed space $(X, \|.,.\|)$ and $l_1, l_2 \in X$.If $I_2\text{-}\lim_{j,k} \|x_{jk} - l_1, z\| = 0$ and $I_2\text{-}\lim_{j,k} \|x_{jk} - l_2, z\| = 0$ then $l_1 = l_2$.*

proof:*Let $l_1 \neq l_2$,hence there exists $z \in X$ such that $0 \neq l_1 - l_2$ and z are linearly independent.Put*

$$\|l_1 - l_2, z\| = 2\varepsilon, with \varepsilon > 0$$

Now

$$2\varepsilon = \|l_1 - x_{jk} + x_{jk} - l_2, z\| \leq \|x_{jk} - l_1, z\| + \|x_{jk} - l_2, z\|$$

Therefor

$$\{(j,k) : \|x_{jk} - l_2, z\| < \varepsilon\} \subseteq \{(j,k) : \|x_{jk} - l_1, z\| \geq \varepsilon\} \in I$$

Hence $\{(j,k) : \|x_{jk} - l_2, z\| < \varepsilon\} \in I$ that is contradict with nontrivial I.

Corollary 3.9. *If $(x_{jk})_{j,k\in\mathbb{N}}$,$(y_{jk})_{j,k\in\mathbb{N}}$ be double sequences in 2-normed space $(X, \|.,.\|)$ and $I_2\text{-}\lim_{j,k} x_{jk} = a$,$I_2\text{-}\lim_{j,k} y_{jk} = b$ then*
(i) $I_2\text{-}\lim_{j,k} x_{jk} + y_{jk} = a + b$

(ii) $I_2\text{-}\lim_{j,k} \alpha x_{jk} = \alpha a$ *,where* $\alpha \in \mathbb{R}$

proof(i):*Let* $\varepsilon > 0$. *For each nonzero* $z \in X$ *we have*

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|(x_{mn}+y_{mn})-(a+b), z\| \geq \varepsilon\} \subseteq \left(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|x_{mn}-a, z\| \geq \frac{\varepsilon}{2}\}\right.$$

$$\left.\bigcup\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|y_{mn}-b, z\| \geq \tfrac{\varepsilon}{2}\}\right) \in I$$

Hence $\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|(x_{mn}+y_{mn})-(a+b), z\| \geq \varepsilon\} \in I$ *and the statements is follows.*

(ii) *The statement is an easy consequence of* (i)

Condition (AP2) In 2-Normed Spaces

In [5], it was proved that I and I^*-convergence of double sequence are equivalent if and only if $I \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ satisfies the condition (AP2).It seems therefore reasonable to consider the condition (AP2) of I_2 and I_2^*-convergence of double sequences in 2-normed spaces.

Definition 3.10. An admissible ideal $I \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ satisfies the condition $(AP2)$ if for every countable family of disjoint sets $\{A_1, A_2, ...\}$ belonging to I ,there exists a countable family of sets $\{B_1, B_2, ...\}$ such that $A_j \bigtriangleup B_j \in I_0$ i.e
$A_j \bigtriangleup B_j$ is included in the finite union of rows and columns $\mathbb{N}\times\mathbb{N}$ for each $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in I$ (hence $B_j \in I$ for each $j \in \mathbb{N}$).

Theorem 4.3: If $(X, \|.,.\|)$ be a finite dimensional 2-normed space and $I \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ be an admissible ideal that satisfied the condition $(AP2)$, then for an arbitrary double sequence $(x_{mn})_{m,n\in\mathbb{N}}$ of elements of X,

$$I_2 - \lim_{m,n} x_{mn} = a \text{ implies } I_2^* - \lim_{m,n} x_{mn} = a.$$

proof: If $I_2 - \lim_{m,n} x_{mn} = a$ then for any $\varepsilon > 0$ and nonzero $z \in X$,

$$A(\varepsilon) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|x_{mn} - a\|_\infty \geq \varepsilon\} \in I$$

Put $A_1 = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|x_{mn} - a\|_\infty \geq 1\}$ and for $k \geq 2$
$A_k = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{k} \leq \|x_{mn} - a\|_\infty < \frac{1}{k-1}\}$ for each nonzero $z \in X$.
Clearly $A_i \bigcap A_j = \emptyset$ for $i \neq j$ and $A_i \in I$ for each $i \in \mathbb{N}$.Hence there exists a countable family of sets $\{B_1, B_2, ...\}$ such that $A_j \Delta B_j$ is included in finite union of rows and columns in $\mathbb{N}\times\mathbb{N}$ for each j and $B = \bigcup_{j=1}^{\infty} B_j \in I$.
Put $M = \mathbb{N}\times\mathbb{N} - B$ and prove $\lim_{mn} x_{mn} = a$ for $(m,n) \in M$.
Let $\delta > 0$ .Choose $k \in \mathbb{N}$ such that $\frac{1}{k} < \delta$. Then

$$\{(m,n) : \|x_{mn} - a\|_\infty \geq \delta\} \subseteq \bigcup_{j=1}^{n} A_j.$$

Since $A_j \Delta B_j$,$j = 1, 2, ..., k$ are included in finite union of rows and columns, there exists $n_0 \in \mathbb{N}$ such that
$(\bigcup_{j=1}^{k} B_j) \bigcap \{(m,n) : m \geq n_0, n \geq n_0\}$

$$= (\bigcup_{j=1}^{k} A_j) \bigcap \{(m,n) : m \geq n_0, n \geq n_0\}$$

If $m, n \geq n_0$ and $(m,n) \notin B$ then

$$(m,n) \notin \bigcup_{j=1}^{k} B_j \text{ and so } (m,n) \notin \bigcup_{j=1}^{k} A_j.$$

This implies that

$$\|x_{mn} - a, z\| < \tfrac{1}{k} < \delta \text{ for each } z \in X.$$

Hence $\lim_{m,n} x_{mn} = a$ for $(m,n) \in M$.

Theorem 4.4: Let $(X, \|.,.\|)$ be 2-normed space.If X has at least one accumulation point for any arbitrary double sequence $(x_{mn})_{m,n\in\mathbb{N}} \subseteq X$ and for each $a \in X$,$I_2 - \lim_{m,n} x_{mn} = a$ implies $I_2^* - \lim_{m,n} x_{mn} = a$,then I has the property (AP2).

proof: Let $a \in X$ is an accumulation point of X.let $(b_k)_{k\in\mathbb{N}}$ be a sequence of distinct elements of X such that $b_k \neq a$ for any k,

$$b_k \xrightarrow{\|.,.\|_X} a$$

Put $\varepsilon_k^z = \|b_k - a, z\|$ for $k \in \mathbb{N}$.Let $(A_j)_{j\in\mathbb{N}}$ be a disjoint family of nonempty sets from I.Define a sequence $(x_{mn})_{m,n\in\mathbb{N}}$ for all $z \in X$

a if $(m,n) \notin A_j$

Let $\delta > 0$,$z_0 \in X$.Choose $k \in \mathbb{N}$ such that $\varepsilon_k^{(z_0)} < \delta$.
Then

$$A^{z_0}(\delta) = \{(m,n) : \|x_{mn} - a, z_0\| \geq \delta\} \subseteq A_1 \bigcup A_2 \bigcup ... \bigcup A_k$$

Hence $A^{z_0}(\delta) \in I$ and so $I_2 - \lim_{m,n} x_{mn} = a$.
By definition(.....)$I_2^* - \lim_{m,n} x_{mn} = a$ and so there exist a set $B \in I$ such that $M = \mathbb{N} \times \mathbb{N} - B \in F(I)$ and

$$x_{mn} \xrightarrow{\|.,.\|_X} a \qquad\qquad (m,n) \in M \tag{3.1}$$

Put $B_j = A_j \bigcap B$ for $j \in \mathbb{N}$.Then $B_j \in I$ for each $j \in \mathbb{N}$.
Moreover $\bigcup_{j=1}^{\infty} B_j = B \bigcap \bigcup_{j=1}^{\infty} A_j \subseteq B$ and so $\bigcup_{j=1}^{\infty} B_j \in I$. Fix $j \in \mathbb{N}$.If $A_j \bigcap M$ is not included in the finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$,then M must contain an infinite sequence of element (m_k, n_k) and $\lim x_{m_k,n_k} = b_j \neq a$ for all $k \in \mathbb{N}$ which it contradicts with (4.1).Hence $A_j \bigcap M$ must be contained in the finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$.
Hence $A_j \Delta B_j = A_j - B_j = A - B = A_j \bigcap M$ is also included in the finite union of rows and columns.This proves that the ideal I has the property (AP2)

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