Int. J. Contemp. Math. Sciences, Vol. 6, 2011, no. 44, 2201 - 2206

Factorable Matrix Transforms of Summability Domains of Cesàro Matrices

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Abstract

In this paper some classes of triangular factorable matrices, transforming the summability domain of Cesàro matrix into the summability domain of a matrix B with real or complex entries, are described.

Mathematics Subject Classification: 40C05, 40G05

Keywords: Matrix transforms, Factorable matrices, Summability method of Cesàro

1 Introduction

In the present paper the transforms of summability domains of Cesàro matrices by triangular factorable matrices are studied. Let $A = (a_{nk})$ be a matrix with real or complex entries. Throughout this paper we assume that indices and summation indices run from 0 to ∞ unless otherwise specified. A sequence $x := (x_k)$ or a series $x := \sum_k x_k$ is said to be A-*summable* if the sequence $Ax = (A_n x)$ is convergent, where

$$A_n x := \sum_k a_{nk} x_k.$$

Let

$$c := \left\{ x = (x_k) \,|\, \exists \lim_k x_k \right\}, \;\; cs := \left\{ x = (x_k) \,|\, \exists \lim_n \sum_{k=0}^{n} x_k \right\}$$

$$l := \left\{ x = (x_k) \,|\, \sum_k |x_k| < \infty \right\}, \;\; c_A := \{ x = (x_k) \,|\, Ax \in c \}.$$

A matrix A is called *sequence-to-sequence conservative* (shortly, *Sq-Sq conservative*) if $Ax \in c$ for each $x \in c$. If $Ax \in c$ for each $x \in cs$, then a matrix

A is called *series-to-sequence conservative* (shortly, *Sr-Sq conservative*). A matrix A is said to be *series-to-sequence regular* (shortly, *Sr-Sq regular*) if $\lim_n A_n x = \lim_n \sum_{k=0}^{n} x_k$ for every $x \in cs$.

Let $\mathcal{M}$ be the set of all lower triangular factorable matrices $M = (m_{nk})$, where

$$m_{nk} = r_n v_k, \ k \leq n; \ \ r_n, v_k \in \mathcal{C},$$

Let $C^\alpha = (a_{nk})$, $\alpha \in \mathcal{C}\backslash\{-1, -2, ...\}$, be a series-to-sequence Cesàro matrix, i.e. (see [4] or [5])

$$a_{nk} := \begin{cases} \frac{A^\alpha_{n-k}}{A^\alpha_n} & (k \leq n), \\ 0 & (k > n), \end{cases}$$

where $A^\alpha_n = \binom{n+\alpha}{n}$ are Cesàro numbers. In [1] and [8] necessary and sufficient conditions for a matrix M with real or complex entries to be a transform from c_{C^α} into c_B for certain $\alpha \in \mathcal{C}$ and certain triangular matrix B are described. Moreover, in [3] this problem is considered for the special case $B = C^\beta$, and in [2] one class of triangular matrices M, transforming c_{C^α} into c_{C^β}, is described.

In the present paper some classes of triangular factorable matrices M, transforming c_{C^α} into c_{C^β}, are described. The paper is organized as follows. In Section 2 some auxiliary results are presented, which are needed later. In Section 3 sufficient conditions for $M \in \mathcal{M}$ to be a transform from c_A into c_B are found. In Section 4 some classes of triangular factorable matrices M from $\mathcal{M}$, transforming c_{C^α} into c_B are described.

2 Auxiliary results

In this section we present some auxiliary results, which we need further.

Lemma 2.1 *(cf. [5], p. 46-47). A matrix $D = (d_{nk})$ is Sq-Sq conservative if and only if*

$$there\, exist\, finite\, limits \ \ \lim_n d_{nk} = d_k, \tag{1}$$

$$there\, exist\, finite\, limits \ \ \lim_n \sum_k d_{nk} = d, \tag{2}$$

$$\sum_k |d_{nk}| = \mathcal{O}(1). \tag{3}$$

Also we need the following properties of Cesàro numbers (see [4], p. 77-81):

$$\sum_{n=k}^{\infty} \frac{A^\alpha_{n-k}}{A^\beta_n} = \frac{\beta}{\beta - \alpha - 1} \frac{1}{A^{\beta-\alpha-1}_k} \text{ for } \ Re\beta \geq 0, \ Re(\beta - \alpha) > 1, \ k = 1, 2, ..., \tag{4}$$

$$|A^\alpha_n| \geq L(n+1)^{Re\alpha} \text{ for } \ \alpha \in \mathcal{C}\backslash\{-1, -2, ...\}, \ L > 0. \tag{5}$$

Lemma 2.2 *(cf. [4], p. 192). Let $\alpha \in \mathcal{C}$ with $Re\alpha > 0$ or $\alpha = 0$, and (v_k) is a sequence of complex numbers. A series $\Sigma_k v_k x_k$ is convergent for each $\sum_k x_k \in c_{C^\alpha}$ if and only if*

$$v_k = \mathcal{O}\left[(k+1)^{-Re\alpha}\right] \tag{6}$$

and

$$\sum_{k=0}^{\infty}(k+1)^{Re\alpha}\left|\Delta_k^{\alpha+1}v_k\right| = \mathcal{O}(1), \tag{7}$$

where

$$\Delta^{\alpha+1}v_k := \sum_{n=k}^{\infty} A_{n-k}^{-\alpha-2}v_n.$$

3 Matrix transforms from c_A into c_B

At first we give a simple necessary condition for $M \in \mathcal{M}$ to be a transform from c_A into c_B.

Proposition 3.1 *Let $A = (a_{nk})$ be a matrix with $e^0 = (1, 0, 0, ...) \in c_A$ and $B = (b_{nk})$ an arbitrary matrix with real or complex entries. If $M = (r_n v_k) \in \mathcal{M}$ transforms c_A into c_B, then $(r_n) \in c_B$.*

Proof easily follows from the relation

$$M_n e^0 = r_n v_0.$$

Now we present sufficient conditions for $M \in \mathcal{M}$ to be a transform from c_A into c_B.

Theorem 3.2 *Let $A = (a_{nk})$ and $B = (b_{nk})$ be matrices with real or complex entries, (r_n) and (v_k) sequences with real or complex entries and $B^t = (b_{pn}^t)$ a matrix, defined by the relation $b_{pn}^t = b_{pn} r_n$. Then $M = (r_n v_k) \in \mathcal{M}$ transforms c_A into c_B if*

$$(v_k x_k) \in cs \; for \; every \; x \in c_A, \tag{8}$$

$$B^t \; is \; Sq - Sq \; conservative. \tag{9}$$

Proof easily follows from the equality

$$\sum_n b_{pn} M_n x = \sum_n b_{pn}^t \sum_{k=0}^{n} v_k x_k$$

for each $x \in c_A$.

Proposition 3.3 *Let $B = (b_{nk})$ be a Sr-Sq regular matrix, where $b_{nk} > 0$ for all n and k, and (r_n) a sequence with real or complex entries. Then condition (9) is satisfied, i.e. $B^t = (b^t_{pn}) = (b_{pn} r_n)$ is Sq-Sq conservative if and only if $(r_n) \in l$.*

Proof. Necessity. We suppose that B^t is Sq-Sq conservative and show that then $(r_n) \in l$. Indeed, condition (3) of Lemma 2.1 takes for $D = B^t$ the form

$$T_p := \sum_n |b_{pn} r_n| = \sum_n b_{pn} |r_n| = \mathcal{O}(1). \tag{10}$$

If $\sum_n |r_n| = \infty$, then (see [6], p. 92) $\lim_{p\to\infty} T_p = \infty$, i.e. condition (10) is not satisfied. Hence $(r_n) \in l$ by Lemma 1.

Sufficiency. Let $(r_n) \in l$. We show that all conditions of Lemma 2.1 are fulfilled for $D = B^t$. Indeed, the Sr-Sq regularity of B implies that $(r_n) \in c_B$, i.e. condition (2) of Lemma 2.1 is satisfied for $D = B^t$. The Sr-Sq regularity of B also implies that $b_{nk} = \mathcal{O}(1)$ and there exist the finite limits $\lim_n b_{nk}$ by Proposition 17 of [7]. Consequently condition (1) is fulfilled for $D = B^t$, and

$$T_p = \mathcal{O}(1) \sum_n |r_n| = \mathcal{O}(1),$$

i.e. condition (3) of Lemma 2.1 is satisfied for $D = B^t$. Therefore B^t is Sq-Sq conservative by Lemma 2.1.

Remark. The assertion of Proposition 3.3 holds also for lower triangular matrix $B = (b_{nk})$, where $b_{nk} > 0$ for all $k \le n$.

Theorem 3.4 *Let $A = (a_{nk})$, $B = (b_{nk})$ be matrices with real or complex entries and (r_n), (v_k) sequences with real or complex entries. Moreover, let $l \subset c_B$ and $(r_n) \in l$. Then $M = (r_n v_k) \in \mathcal{M}$ transforms c_A into c_B if condition (8) is fulfilled.*

Proof. Let

$$S_n := \sum_{k=0}^{n} v_k x_k$$

for every $x \in c_A$. As $(S_n) \in c$ for every $x \in c_A$ by (8), then (S_n) is also bounded for each $x \in c_A$. Therefore

$$\sum_n |M_n x| = \sum_n |r_n S_n| = \mathcal{O}(1) \sum_n |r_n| = \mathcal{O}(1)$$

for every $x \in c_A$. As $l \subset c_B$, then M transforms c_A into c_B.

4 Matrix transforms from c_{C^α} into c_B

In this section we consider the factorable matrix transforms of summability domains of Cesàro matrices.

Theorem 4.1 *Let $\alpha \in \mathcal{C}$ with $Re\alpha > 0$ or $\alpha = 0$, and $B = (b_{nk})$ be a matrix with the property $l \subset c_B$. Let (v_k) be defined by $v_k := 1/A_k^t$, where $t \in \mathcal{C}$ with $Ret > 0$, and $(r_n) \in l$. Then $M = (r_n v_k) \in \mathcal{M}$ transforms c_{C^α} into c_B if $Re\alpha \leq Ret$.*

Proof. By Theorem 3.4 it is sufficient to show that condition (8) is fulfilled for $A = C^\alpha$ and $v_k = 1/A_k^t$. With the help of (4) and (5) we have

$$\sum_{k=0}^{\infty}(k+1)^{Re\alpha}\left|\Delta_k^{\alpha+1}v_k\right| = \sum_{k=0}^{\infty}(k+1)^{Re\alpha}\left|\sum_{n=k}^{\infty}\frac{A_{n-k}^{-\alpha-2}}{A_n^t}\right|$$

$$= \sum_{k=0}^{\infty}(k+1)^{Re\alpha}\left|\frac{t}{t+\alpha+1}\frac{1}{A_k^{t+\alpha+1}}\right| = \mathcal{O}(1)\sum_{k=0}^{\infty}\frac{(k+1)^{Re\alpha}}{(k+1)^{Re(t+\alpha)+1}}$$

$$= \mathcal{O}(1)\sum_{k=0}^{\infty}\frac{1}{(k+1)^{Re(t+1)}} = \mathcal{O}(1),$$

since $Ret > 0$, i.e. condition (7) is satisfied. Condition (6) is also fulfilled, since by (5) there exists $L > 0$ so that

$$\left|\frac{1}{A_k^t}\right| \leq \frac{1}{L(k+1)^{Ret}} = \mathcal{O}(1)k+1)^{-Ret} = \mathcal{O}(1)k+1)^{-Re\alpha}.$$

Consequently condition (8) is fulfilled by Lemma 2.2. Thus M transforms c_{C^α} into c_B by Theorem 3.4.

Theorem 4.2 *Let $\alpha \in \mathcal{C}$ with $Re\alpha > 0$ or $\alpha = 0$, and $B = (b_{nk})$ be a matrix with the property $l \subset c_B$. Let (v_k) be defined by $v_k := y^k$, where $y \in \mathcal{C}$, and $(r_n) \in l$. Then $M = (r_n v_k) \in \mathcal{M}$ transforms c_{C^α} into c_B if $|y| < 1$.*

Proof. By Theorem 3.4 it is sufficient to show that condition (8) is fulfilled for $A = C^\alpha$ and $v_k = y^k$. As

$$\sum_{k=0}^{\infty}(k+1)^{Re\alpha}\left|\Delta_k^{\alpha+1}v_k\right| = \sum_{k=0}^{\infty}(k+1)^{Re\alpha}\left|\sum_{n=0}^{\infty}A_n^{-\alpha-2}y^{n+k}\right|$$

$$\leq \sum_{k=0}^{\infty}(k+1)^{Re\alpha}y^k\sum_{n=0}^{\infty}\left|A_n^{-\alpha-2}y^n\right| = \mathcal{O}(1)\sum_{k=0}^{\infty}(k+1)^{Re\alpha}y^k < \infty$$

(since the series $\sum_{k=0}^{\infty}(k+1)^{Re\alpha}y^k$ converges by the convergence criterion of Cauchy for positive series), then condition (7) is satisfied. Also condition (6) is fulfilled, since

$$\lim_k y^k(k+1)^{Re\alpha} = 0.$$

Consequently condition (8) is fulfilled. Thus M transforms c_{C^α} into c_B by Theorem 3.4.

ACKNOWLEDGEMENTS. This work was supported by Estonian Science Foundation grant 8627.

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Received: April, 2011, 200x