

M-Projective Curvature Tensor on Kaehler Manifold

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Abstract

Properties of M – projective curvature tensor been studied on Kaehler manifold with respect to recurrent and symmetric properties.

1. Introduction

We consider a $2n$ – dimensional Kaehler manifold M_{2n} with a vector valued linear function F and a Riemannian metric g which satisfies the following conditions:

1.1) $\bar{X} = -I_{2n}$, where $F(X) = \bar{X}$.

1.2) $g(\bar{X}, \bar{Y}) = g(X, Y)$.

1.3) ${}^1F(X, T) = g(\bar{X}, Y)$.

1.4) $(D_X F)Y = 0$, where D is the Riemannian connection.

If we define [1]

1.5) a) ${}^1H(Y, Z) = -\frac{1}{2} C_3^1 \overline{R(Y, Z)}$, **b)** $*H(Y, Z) = -{}^1H(Y, \bar{Z})$

Then we have

1.6) a) ${}^1H(Y, Z) = S(Y, \bar{Z})$, **b)** $*H(Y, Z) = -{}^1H(Y, \bar{Z}) = S(Y, Z)$

where R and S are the so called Riemannian curvature tensor and the Ricci tensor respectively.

The projective curvature tensor W , Conformal curvature tensor C , Conharmonic curvature tensor L , Conircular curvature tensor V , H -projective curvature tensor P , H -conharmonic curvature tensor T , H -Conircular curvature tensor K , Conharmonic* curvature tensor T^* , H -conformal (Bochner) curvature tensor B and the conformal* curvature tensor C^* are given on Kaehler manifold respectively by;

$$1.7) a) W(X, Y, Z) = R(X, Y, Z) - \frac{1}{2n-1} [S(Y, Z)X - S(X, Z)Y]$$

$$b) C(X, Y, Z) = R(X, Y, Z) - \frac{1}{2(n-1)} [S(Y, Z)X - S(X, Z)Y - g(X, Z)RY + g(Y, Z)RX] + \frac{r}{2(n-1)(2n-1)} [g(Y, Z)X - g(X, Z)Y]$$

$$c) L(X, Y, Z) = R(X, Y, Z) - \frac{1}{2(n-1)} [S(Y, Z)X - S(X, Z)Y - g(X, Z)RY + g(Y, Z)RX]$$

$$d) V(X, Y, Z) = R(X, Y, Z) - \frac{r}{2n(2n-1)} [g(Y, Z)X - g(X, Z)Y]$$

$$e) P(X, Y, Z) = R(X, Y, Z) - \frac{1}{2(n+1)} [S(Y, Z)X - S(X, Z)Y + S(X, \bar{Z})\bar{Y} - S(Y, \bar{Z})\bar{X} + 2S(X, \bar{Y})\bar{Z}]$$

$$f) T(X, Y, Z) = R(X, Y, Z) - \frac{1}{2(n+2)} [S(Y, Z)X - S(X, Z)Y + S(\bar{Y}, Z)\bar{X} - S(\bar{X}, Z)\bar{Y} + 2S(\bar{X}, Y)\bar{Z} + g(Y, Z)RX - g(X, Z)RY + {}^1F(Y, Z)R\bar{X} - {}^1F(X, Z)R\bar{Y} - 2{}^1F(X, Y)R\bar{Z}]$$

$$g) K(X, Y, Z) = R(X, Y, Z) - \frac{r}{4n(n+1)} [g(Y, Z)X - g(X, Z)Y + {}^1F(Y, Z)\bar{X} - {}^1F(X, Z)\bar{Y} - 2{}^1F(X, Y)\bar{Z}]$$

$$h) T^*(X, Y, Z) = R(X, Y, Z) + \frac{r}{2(n-1)(2n-1)} [g(Y, Z)X - g(X, Z)Y]$$

$$i) B(X, Y, Z) = R(X, Y, Z) - \frac{1}{2(n+2)} [S(Y, Z)X - S(X, Z)Y - g(X, Z)RY + g(Y, Z)RX + S(X, \bar{Z})\bar{Y} - S(Y, \bar{Z})\bar{X} - {}^1F(X, Z)R\bar{Y} + {}^1F(Y, Z)R\bar{X} - 2{}^1F(X, Y)R\bar{Z} + 2S(X, \bar{Y})\bar{Z}]$$

$$\begin{aligned}
 & + \frac{r}{4(n+1)(n+2)} [g(X, Z)Y - g(Y, Z)X - {}^1F(X, Z)\bar{Y} \\
 & + {}^1F(Y, Z)\bar{X} - {}^1F(X, Y)\bar{Z}] \\
 j) \quad C^*(X, Y, Z) = & R(X, Y, Z) - \frac{1}{2(n-1)} [g(Y, Z)RX - g(X, Z)RY] \\
 & + \frac{r}{2(n-1)(2n-1)} [g(Y, Z)X - g(X, Z)Y]
 \end{aligned}$$

where r is the scalar curvature tensor.

A Kaehler manifold is said to be recurrent if for a non-zero recurrence vector v , that satisfies.

1.8) $(D_U R)(X, Y, Z) = v(U)R(X, Y, Z)$, from which we have.

1.9) a) $(D_U S)(Y, Z) = v(U)S(Y, Z)$

b) $(D_U R)(Y, Z) = v(U)RY$, where $g(RY, Z) = S(Y, Z)$, and

1.10) $D_U r = v(U)r$

Also on a Kaehler manifold 1H is said to be recurrent if it satisfies.

1.11) $(D_U {}^1H)(X, Y) = v(U) {}^1H(X, Y)$

A Kaehler manifold is said to be symmetric if it satisfies.

1.12) $(D_U R)(X, Y, Z) = 0$

From which we have

1.13) a) $(D_U S)(Y, Z) = 0$

b) $(D_U R)Y = 0$, and

1.14) $D_U r = 0$

If Q stands for the curvature tensor vide equations **a-j** of **1.7** then it is said that a Kaehler manifold is Q -recurrent if it satisfies.

1.15) $(D_U Q)(X, Y, Z) = v(U)Q(X, Y, Z)$

for a non-zero recurrence vector v .

And it is said that a Kaehler manifold is Q -symmetric If it satisfies.

1.16) $(D_U Q)(X, Y, Z) = 0$

The M -projective curvature on tensor on a Kaehler manifold is given by:

1.17) $M(X, Y, Z) = R(X, Y, Z) - \frac{1}{4(n-1)} [S(Y, Z)X - S(X, Z)Y$
 $- S(Y, \bar{Z})\bar{X} + S(X, \bar{Z})\bar{Y}]$

2. M-projective recurrent Kaehler manifold

A Kaehler manifold is said to be M – projective recurrent if it satisfies.

$$2.1) (D_U M)(X, Y, Z) = \nu(U)M(X, Y, Z)$$

For a non zero recurrence vector ν .

From 1.17 we have;

$$2.2) (D_U M)(X, Y, Z) - \nu(U)M(X, Y, Z) = (D_U R)(X, Y, Z)$$

$$- \nu(U)R(X, Y, Z) - \frac{1}{4(n-1)} [((D_U S)(Y, Z) - \nu(U)S(Y, Z))X$$

$$- ((D_U S)(X, Z) - \nu(U)S(X, Z))Y - ((D_U S)(Y, \bar{Z})$$

$$- \nu(U)S(Y, \bar{Z}))\bar{X} + ((D_U S)(X, \bar{Z}) - \nu(U)S(X, \bar{Z}))Y]$$

If the manifold is M – projective recurrent we have;

$$2.3) (D_U R)(X, Y, Z) - \nu(U)R(X, Y, Z) - \frac{1}{4(n-1)} [((D_U S)(Y, Z)$$

$$- \nu(U)S(Y, Z))X - ((D_U S)(X, Z) - \nu(U)S(X, Z))Y$$

$$- ((D_U S)(Y, \bar{Z}) - \nu(U)S(Y, \bar{Z}))\bar{X} + ((D_U S)(X, \bar{Z})$$

$$- \nu(U)S(X, \bar{Z}))Y] = 0$$

Contracting this equation with respect to X we get;

$$2.4) \frac{n-2}{2(n-1)} ((D_U S)(Y, Z) - \nu(U)S(Y, Z)) = 0$$

Hence we can state:

Theorem 2.1: A M – projective recurrent Kaehler manifold M_{2n} , $n > 2$ is Ricci recurrent.

Theorem 2.2: A Kaehler manifold M_{2n} , $n > 2$ M – projective recurrent if and only if it is Ricci recurrent.

Theorem 2.3: A Flat Kaehler manifold is M – projective recurrent if and only if it is Ricci recurrent.

Theorem 2.4: If on a Kaehler manifold two of the following hold, the third also hold.

- a. It is M – projective recurrent manifold.
- b. It is Ricci recurrent manifold.
- c. It is M – projective recurrent manifold.

Now barring Z in 2.4 and using 1.6.a we get:

$$2.5) \frac{n-2}{2(n-1)} ((D_U {}^1H)(Y, Z) - \nu(U) {}^1H(Y, Z)) = 0$$

Hence, we have;

Theorem 2.5: On an M –projective recurrent Kaehler manifold M_{2n} , $n > 2$, 1H is recurrent.

From **1.17** and **1.7.d** we have;

$$2.6) M(X, Y, Z) = V(X, Y, Z) - \frac{1}{4(n-1)} [S(Y, Z)X - S(Y, Z)Y - S(Y, \bar{Z})\bar{X} + S(X, \bar{Z})\bar{Y}] + \frac{r}{2n(2n-1)} [g(Y, Z)X - g(X, Z)Y]$$

From which we have:

$$2.7) (D_U M)(X, Y, Z) - \nu(U)M(X, Y, Z) = (D_U V)(X, Y, Z) - \nu(U)V(X, Y, Z) - \frac{1}{4(n-1)} [((D_U S)(Y, Z) - \nu(U)S(Y, Z))X - ((D_U S)(X, Z) - \nu(U)S(X, Z))Y - ((D_U S)(Y, \bar{Z}) - \nu(U)S(Y, \bar{Z}))\bar{X} + ((D_U S)(X, \bar{Z}) - \nu(U)S(X, \bar{Z}))\bar{Y}] + \frac{(D_U r - \nu(U)r)}{2n(2n-1)} [g(Y, Z)X - g(X, Z)Y]$$

Hence, we can state:

Theorem 2.6: On a Kaehler manifold if any two of the following hold, the third also hold:

- a. It is M-projective recurrent manifold.
- b. It is Conircular recurrent manifold.
- c. It is a Ricci recurrent manifold.

Similarly we can prove nine theorems analog to theorem 2.6 by simply replacing Conircular in part b of the theorem by projective, conformal, conharmonic, H –projective H –conformal, H –conharmonic, H –Conircular, Conharmonic C^* , and Conformal*.

Now from **1.7.j** we can have:

$$2.8) (D_U C^*)(X, Y, Z) - \nu(U)C^*(X, Y, Z) = (D_U R)(X, Y, Z) - \nu(U)R(X, Y, Z) - \frac{1}{2(n-1)} [g(Y, Z)((D_U R)X - (D_U R)Y) - g(X, Z)((D_U R)Y - \nu(U)RY)] + \frac{(D_U r - \nu(U)r)}{2(n-1)(2n-1)} [g(Y, Z)X - g(X, Z)Y]$$

If the manifold is Conformal* recurrent we have;

$$\begin{aligned}
2.9) & (D_U R)(X, Y, Z) - \nu(U)R(X, Y, Z) \\
& - \frac{1}{2(n-1)} [g(Y, Z)((D_U R)X - \nu(U)RX)] \\
& - g(X, Z)((D_U R)Y - \nu(U)RY) \\
& + \frac{(D_U r - \nu(U)r)}{2(n-1)(2n-1)} [g(Y, Z)X - g(X, Z)Y] = 0
\end{aligned}$$

Contracting this equation with respect to X we get:

$$2.10) \quad \frac{2(n-1)}{2n-2} ((D_U S)(Y, Z) - \nu(U)S(Y, Z)) = 0$$

Hence a Conformal* recurrent Kaehler manifold is Ricci recurrent.

But from 1.17 and 1.7.j we have;

$$\begin{aligned}
2.11) \quad M(X, Y, Z) &= C^*(X, Y, Z) - \frac{1}{4(n-1)} [S(Y, Z)X \\
& - S(X, Z)Y - S(Y, \bar{Z})\bar{X} + S(X, \bar{Z})\bar{Y}] - \frac{1}{2(n-1)} [g(Y, Z)RX \\
& - g(X, Z)RY] + \frac{r}{2(n-1)(2n-1)} [g(Y, Z)X - g(X, Z)Y]
\end{aligned}$$

From which we can get;

$$\begin{aligned}
2.12) \quad (D_U M)(X, Y, Z) - \nu(U)M(X, Y, Z) &= (D_U C^*)(X, Y, Z) \\
& - \nu(U)C^*(X, Y, Z) - \frac{1}{4(n-1)} [((D_U S)(Y, Z) - \nu(U)S(Y, Z))X \\
& - ((D_U S)(X, Z) - \nu(U)S(X, Z))Y - ((D_U S)(Y, \bar{Z}) - \nu(U)S(Y, \bar{Z}))\bar{X} \\
& + ((D_U S)(X, \bar{Z}) - \nu(U)S(X, \bar{Z}))\bar{Y}] - \frac{1}{2(n-1)} [g(Y, Z)((D_U R)X \\
& - \nu(U)RX) - g(X, Z)((D_U R)Y - \nu(U)RY)] + \frac{(D_U r - \nu(U)r)}{2(n-1)(2n-1)} [g(Y, Z)X \\
& - g(X, Z)Y].
\end{aligned}$$

Therefore, we have in consequence of theorem 2.1 and equations 2.10 & 2.9.

Theorem 2.7: A Kaehler manifold $M_{2n}, n > 2$ is M -projective recurrent if and only if it is Conformal* recurrent.

Now from 1.7.d we have:

$$\begin{aligned}
2.13) \quad (D_U V)(X, Y, Z) - \nu(U)V(X, Y, Z) &= (D_U R)(X, Y, Z) \\
& - \nu(U)R(X, Y, Z) - \frac{(D_U r - \nu(U)r)}{2n(2n-1)} [g(Y, Z)X - g(X, Z)Y]
\end{aligned}$$

If the manifold Concircular recurrent we have:

$$2.14) (D_U R)(X, Y, Z) - v(U)R(X, Y, Z) = \frac{(D_U r - v(U)r)}{2n(2n-1)} [g(Y, Z)X - g(X, Z)Y]$$

Contracting this equation with respect to X we obtain;

$$2.15) (D_U S)(Y, Z) - v(U)S(Y, Z) = \frac{(D_U r - v(U)r)}{2n} g(Y, Z)$$

If $r = 0$, we have; $(D_U S)(Y, Z) - v(U)S(Y, Z) = 0$, which means that the manifold is Ricci-recurrent. But from 1.17 and 1.7.d we can have:

$$2.16) M(X, Y, Z) = V(X, Y, Z) - \frac{1}{4(n-1)} [S(Y, Z)X - S(X, Z)Y - S(Y, \bar{Z})\bar{X} + S(X, \bar{Z})\bar{Y}] + \frac{r}{2n(2n-1)} [g(Y, Z)X - g(X, Z)Y]$$

Form which we can get:

$$2.17) (D_U M)(X, Y, Z) - v(U)M(X, Y, Z) - (D_U V)(X, Y, Z) - v(U)V(X, Y, Z) - \frac{1}{4(n-1)} [((D_U S)(Y, Z) - v(U)S(Y, Z))X - ((D_U S)(X, Z) - v(U)S(X, Z))Y - ((D_U S)(Y, \bar{Z}) - v(U)S(Y, \bar{Z}))\bar{X} + ((D_U S)(X, \bar{Z}) - v(U)S(X, \bar{Z}))\bar{Y}] + \frac{(D_U r - v(U)r)}{2n(2n-1)} [g(Y, Z)X - g(X, Z)Y]$$

Hence, we have:

Theorem 2.8: A necessary and sufficient condition for a Kaehler manifold M_{2n} , $n > 2$ of zero scalar curvature to be M – projective recurrent is that it is Concircular recurrent manifold.

Similarly we can prove :

Theorem 2.9: A necessary and sufficient condition for a Kaehler manifold M_{2n} , $n > 2$ of zero scalar curvature to be M – projective recurrent is that it is H – Concircular manifold.

Theorem 2.10: A necessary and sufficient condition for a Kaehler manifold M_{2n} , $n > 2$ of zero for curvature to be M – projective recurrent is that it is Conharmonic* recurrent manifold.

3. M-projective symmetric Kaehler manifold

A Kaehler manifold is said to be M – projective symmetric if it satisfies.

$$3.1) (D_U M)(X, Y, Z) = 0.$$

It is clear that every symmetric Kaehler manifold is M -projective symmetric.

From 1.17 and 3.1 we have if the manifold is M -projective symmetric,

$$3.2) (D_U R)(X, Y, Z) - \frac{1}{4(n-1)} [(D_U S)(Y, Z)X - (D_U S)(X, Z)Y - (D_U S)(X, \bar{Z})\bar{Y} \\ + (D_U S)(X, \bar{Z})\bar{Y}] = 0$$

Contracting this equation with respect to X we get;

$$3.3) \frac{n-2}{2(n-1)} (D_U S)(Y, Z) = 0.$$

If $n \neq 2$, then the manifold is Ricci-Symmetric. That is, equations 1.13 and 1.14 holds. Hence, we have that,

Theorem 3.1: A necessary and sufficient condition for a M -projective symmetric Kaehler manifold to be symmetric is that it is Ricci-symmetric.

Theorem 3.2: An M -projective symmetric Kaehler manifold, M_{2n} , $n > 2$ is Ricci-symmetric.

Theorem 3.3: Every M -projective symmetric Kaehler manifold M_{2n} , $n > 2$ is symmetric.

Theorem 3.4: On M -projective symmetric Kaehler manifold M_{2n} , $n > 2$, the scalar curvature is constant.

Now using Bianchi identity on 3.2 we have:

$$3.4) (D_X R)(Y, Z) - (D_Y R)(X, Z) - \frac{1}{4(n-1)} [(D_U S)(Y, Z)X - (D_U S)(X, Z)Y \\ - (D_U S)(Y, \bar{Z})\bar{X} + (D_U S)(X, \bar{Z})\bar{Y}] = 0$$

Contracting this equation with respect to U we get;

$$3.5) \frac{1}{4(n-1)} [(4n-5)((D_X S)(Y, Z) - (D_Y S)(Y, Z)) + (D_{\bar{X}}^1 H)(Y, Z) - \\ (D_{\bar{Y}}^1 H)(X, Z)] = 0$$

Hence we can state:

Theorem 3.5: On an M -projective symmetric Kaehler manifold we have equation 3.5.

Theorem 3.6: On an M -projective symmetric Kaehler manifold, the first covariant derivative of the Ricci tensor is symmetric if and only if $(D_{\bar{X}}^1 H)(Y, Z) = (D_{\bar{Y}}^1 H)(X, Z)$

Theorem 3.7: Every Einstein M -Projective symmetric Kaehler manifold is symmetric.

Proof: For an Einstein manifold the scalar curvature is constant and the Ricci tensor is given by:

3.6) $S(X,Y) = \frac{r}{2n}g(X,Y)$. Therefore, we have; $(D_U S)(X,Y) = 0$. Hence the statement follows from **3.2**.

Theorem 3.8: Every recurrent M – projective symmetric Kaehler manifold is M – projectively flat.

The proof is obvious.

Theorem 3.9: An M – projective symmetric Kaehler manifold is Ricci-recurrent if and only if;

$$\mathbf{3.7)} \quad (D_U R)(X,Y,Z) + v(U)[M(X,Y,Z) - R(X,Y,Z)] = 0$$

Proof: If the manifold is Ricci recurrent then we have in consequence of **1.9** and **3.2**.

$$\mathbf{3.8)} \quad (D_U R)(X,Y,Z) - \frac{v(U)}{4(n-1)}[S(Y,Z)X - S(X,Z)Y - S(Y,Z)\bar{X} + S(X,Z)\bar{Y}] = 0$$

Using **1.17** and **3.8** we get **3.7**.

Conversely, if **3.7** is true then using it on **3.2** we get; $(D_U S)(Y,Z) = v(U)S(Y,Z)$. Hence we have the statement.

Theorem 3.10: A recurrent Einstein M – projective symmetric space is flat .

Proof: For an Einstein manifold we have:

$$\mathbf{3.9)} \quad (D_U M)(X,Y,Z) = (D_U R)(X,Y,Z).$$

If the manifold is M – projective recurrent we have for a non-zero recurrence vector v . $(D_U M)(X,Y,Z) = v(U)M(X,Y,Z)$. By theorem **2.9** we have; $(D_U M)(X,Y,Z) = v(U)R(X,Y,Z)$. Since the manifold is M -projective symmetric we have; $v(U)R(X,Y,Z) = 0$. Hence we have the statement, since v non-zero.

Now differentiating **2.16** covariant we get;

$$\mathbf{3.10)} \quad (D_U M)(X,Y,Z) = (D_U V)(X,Y,Z) - \frac{1}{4(n-1)}[(D_U S)(Y,Z)X - (D_U S)(X,Z)Y - (D_U S)(Y,\bar{Z})\bar{X} - (D_U S)(X,\bar{Z})\bar{Y}] + \frac{D_U r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y]$$

Therefore we can have in consequence of **1.13**, **1.14**, **1.16** and **3.1**.

Theorem 3.11: On a Kaehler manifold if any two of the following hold, the third also hold.

- a. It is M – projective symmetric manifold .
- b. It is Conircular symmetric manifold.
- c. It is Ricci symmetric manifold.

Similarly we can prove nine theorems analog to theorem 3.10 by simply replacing Concircular in part b of the theorem by projective, conformal, H -conharmonic, H -Concircular, Conharmonic*, and Conformal*.

Theorem 3.12: If an M -projective symmetric Kaehler manifold is Concircular recurrent and Ricci-recurrent under the same recurrence Vector, then it is M -projectively flat.

Proof: Using the facts given in theorem 2.10 we get;

$$\begin{aligned} \mathbf{3.11)} \quad V(X, Y, Z) - \frac{1}{4(n-1)} [S(Y, Z)X - S(X, Z)Y - S(Y, \bar{Z})\bar{X} + S(X, \bar{Z})\bar{Y}] \\ + \frac{r}{2n(2n-1)} [g(Y, Z)X - g(X, Z)Y] = 0 \end{aligned}$$

Hence the result follows from 3.16.

Theorem 3.13: If a Kaehler manifold is Concircular symmetric, M -projective recurrent and Ricci-recurrent under the same recurrent vector, then it is Concircular flat.

The proof is similar to the proof of the above theorem.

Theorem 3.14: If an M -projective symmetric Kaehler manifold is Concircular symmetric and Ricci-recurrent under the same recurrent vector, then the M -projective and the Concircular tensors coincide.

Proof: Using the fact given in theorem 3.10 we get :

$$\begin{aligned} \mathbf{3.12)} \quad -\frac{1}{4(n-1)} [S(Y, Z)X - S(X, Z)Y - S(Y, \bar{Z})\bar{X} + S(X, \bar{Z})\bar{Y}] \\ + \frac{r}{2n(2n-1)} [h(Y, Z)X - g(X, Z)Y] = 0 \end{aligned}$$

Hence the statement follows from 3.9.

Similarly we can prove three theorems analog to the above three theorems by simply replacing Concircular in each one by projective, conformal, conharmonic, H -projective, H -conformal, H -Conharmonic, H -Concircular, Conharmonic*, and Conformal*.

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