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Results on a New Class of Univalent Functions with Two Fixed Points

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Abstract. This paper deals with functions of the form $f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n$, $(a_n \geq 0)$. We introduce a new class M (A, B, z_0, δ, μ) of analytic defined by functions fractional derivatives having two fixed points. We obtain necessary and sufficient condition and a distortion theorem for this new class.

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Introduction & Definitions

Let $\mathcal{T}$ denote the class of functions of the form

$$f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n \text{, where } \quad (a_1 > 0; a_n \geq 0) \tag{1.1}$$

which are analytic and univalent in the unit disc $U = \{z : |z| < 1\}$.

If f (z) and g (z) are any two functions in the class $\mathcal{T}$ such that f (z) is defined by (1.1) and g (z) is defined by

$$g(z) = b_1 z - \sum_{n=2}^{\infty} b_n z^n \text{, where } \quad (b_1 > 0; b_n \geq 0) \tag{1.2}$$

Then Quasi-Hadamard product of f (z) and g (z) is denoted by (f * g) (z), and is defined by the power series

$$(f * g)(z) = a_1 b_1 z - \sum_{n=2}^{\infty} a_n b_n z^n \tag{1.3}$$

A function f (z) belonging to the class $\mathcal{T}$ is said to be starlike function of order α and type β beta if and only if

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} + (1-2\alpha)} \right| < \beta, \qquad z \in U \tag{1.4}$$

For $0 \leq \alpha < 1$ and $0 < \beta \leq 1$. We denote by $\mathcal{T}$ (α , β) the class of all starlike functions of order α and type β, furthermore, a function f (z) belonging to the class $\mathcal{T}$ is said to be convex function of order α and type β if and only if z f ' (z) ∈ $\mathcal{T}$ (α , β) . We denote by $\mathcal{C}$(α, β) the class of all convex functions of order α and type β. The class $\mathcal{T}$ (α, β) and $\mathcal{C}$ (α, β) were studied by Gupta and Ahmad [5] and by Srivastava, Sekine, Owa and Nishimoto [1]. In particular for a_1 = 1, the class $\mathcal{T}$ (α, β) and $\mathcal{C}$ (α, β) were studied by Gupta and Jain [4].

Motivated by Gupta and Ahmad [5] and Kulkarni and Naik [3], we introduce a new class $\mathbb{M}$ (A, B, z_0, δ, μ) of analytic functions defined by fractional derivatives having two fixed points, as defined below:

A function f (z) defined by (1.1) and satisfying

$$\phi(\delta, n) = \frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)} \frac{(1-\mu) f(z_0)}{z_0} + \mu f'(z_0) = 1, 0 < z_0 < 1 \tag{1.5}$$

is said to be in the class $\mathbb{M}$ (A, B, z_0, δ, μ) if

$$\left|\frac{F^{\delta}(z)-F^{(\delta-1)}(z)}{BF^{\delta}(z)-AF^{(\delta-1)}(z)}\right|<1, z\in u \tag{1.6}$$

where $-1\le A<B\le 1,\ 0\le\mu\le 1$ and $F^{\delta}(z)=\Gamma(2-\delta)z^{\delta-1}D_z^{\delta}f(z)$.

Here $D_z^{\delta}f(z)$ denotes the fractional derivative of f (z) of order δ and is defined by

$$D_z^{\delta}f(z)=\frac{1}{\Gamma(1-\delta)}\frac{d}{dz}\int_0^z\frac{f(\zeta)}{(z-\zeta)^{\delta}}d\zeta, \tag{1.7}$$

where $0\le\delta<1,$ and f (z) is an analytic function in a simply-connected region ζ of the z-plane containing the origin.

Thus the condition (1.6) reduces, when $A=(2\alpha-1)\beta,\ B=\beta$ and $\delta=1$, to the inequality (1.4) and we have

$\mathbb{M}((2\alpha-1)\beta, \beta, z_0, 1, 0) = \mathcal{T}_o(\alpha, \beta, z_0)$,

and

$\mathbb{M}((2\alpha-1)\beta, \beta, z_0, 1, 1) = \mathcal{T}_1(\alpha, \beta, z_0)$,

Main Results

In a paper Sharma and Singh [2] obtained sufficient condition for the function f (z) belonging to the class G (λ, μ, A, B, b). This motivates to find necessary and sufficient condition for this new class $\mathbb{M}$(A, B, z_0, δ, μ).

Necessary and sufficient condition:

Theorem (1): A function f (z) defined by (1.1) belongs to the class $\mathbb{M}$(A, B, z_0, δ, μ) if and only if

$$\sum_{n=2}^{\infty}\frac{\phi(\delta,n)}{(n+1-\delta)}\{(1+B)(n-1)+(2\delta)(B-A)\}a_n\le a_1(B-A), \tag{1.8}$$

where $$a_1=1+\sum_{n=2}^{\infty}(1-\mu+\mu n)a_n z_0^{n-1}$$

and $$\phi(\delta,n)=\frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)}$$

Proof: Let $|z|=1$. Then from (1.6), we have

$$\left|F^{\delta}(z)-F^{(\delta-1)}(z)\right|-\left|BF^{\delta}(z)-AF^{(\delta-1)}(z)\right|=\left|-\sum_{n=2}^{\infty}\frac{(n-1)}{(n+1-\delta)}\varphi(\delta,n)a_n z^{n-1}\right|$$

$$-\left|(B-A)a_1+\sum_{n=2}^{\infty}\varphi(\delta,n)\left\{\frac{B(n-1)+(B-A)(2-\delta)}{(n+1-\delta)}\right\}a_n z^{n-1}\right|$$

$$\le \sum_{n=2}^{\infty} \frac{\varphi(\delta,n)}{(n+1-\delta)}\left[(1+B)(n-1)+(2-\delta)(B-A)\right]a_n - a_1(B-A)$$

≤ 0, by hypothesis.

Hence, by maximum modulus theorem, f (z) belongs to the class M (A, B, z_0, δ, μ).

To show the converse, let

$$\sum_{n=2}^{\infty} \frac{\phi(\delta,n)}{(n+1-\delta)}\left\{(1+B)(n-1)+(2-\delta)(B-A)\right\}\left|a_n\right| \le a_1(B-A) < 1, z \in u.$$

Since $\left|\text{Re}(z)\right| \le |z|$ for all z, we have

$$\text{Re}\left\{ \frac{\sum_{n=2}^{\infty} \frac{(n-1)}{(n+1-\delta)}\phi(\delta,n)a_n z^{n-1}}{(B-A)a_1 + \sum_{n=2}^{\infty}\phi(\delta,n)\left\{\frac{B(n-1)+(2-\delta)(B-A)}{(n+1-\delta)}\right\}a_n z^{n-1}} \right\} \quad (1.9)$$

< 1.

Choose values of z on the real axis so that $\frac{F^{\delta}(z)}{F^{(\delta-1)}(z)}$ is real. On clearing the denominator of (1.9) and letting z→1 through real values, we obtain

$$\sum_{n=2}^{\infty} \frac{\phi(\delta,n)}{(n+1-\delta)}\left\{(1+B)(n-1)+(2-\delta)(B-A)\right\}\left|a_n\right| \le a_1(B-A).$$

This completes the proof of the theorem.

Distortion Theorem:

Theorem (2): Let the function f (z) defined by (1.1) belongs to the class M (A, B, z_0, δ, μ). Then

$$a_1\left[|z| - \frac{(B-A)(2-\delta)(3-\delta)}{2\{1+B+(2-\delta)(B-A)\}}|z|^2\right] \le |f(z)| \le a_1\left[|z| + \frac{(B-A)(2-\delta)(3-\delta)}{2\{1+B+(2-\delta)(B-A)\}}|z|^2\right] \quad (2.0)$$

and

$$\frac{a_1}{\Gamma(2-\delta)}\left[|z|^{1-\delta} - \frac{(B-A)(3-\delta)}{\{1+B+(2-\delta)(B-A)}|z|^{2-\delta}\right] \le \left|D_z^{\delta} f(z)\right|$$

$$\le \frac{a_1}{\Gamma(2-\delta)}\left[|z|^{1+\delta} + \frac{(B-A)(3-\delta)}{\{1+B+(2-\delta)(B-A)}|z|^{2-\delta}\right] \quad (2.1)$$

where $$a_1 = 1 + \sum_{n=2}^{\infty} (1 - \mu + \mu n) a_n z_0^{n-1}, \quad z \in u.$$

Proof: In view of equation (1.8) and the fact $\phi(\delta, n)$ is non-decreasing for $n \geq 2$, we have

$$\frac{2}{(2-\delta)(3-\delta)}\{1+B+(2-\delta)(B-A)\}\sum_{n=2}^{\infty} a_n$$
$$\leq \sum_{n=2}^{\infty} \frac{\phi(\delta,n)}{(n+1-\delta)}\{(1+B)(n-1)+(2-\delta)(B-A)\} \leq a_1(B-A). \tag{2.2}$$

which is equivalent to

$$\sum_{n=2}^{\infty} a_n \leq \frac{a_1(B-A)(2-\delta)(3-\delta)}{2\{1+B+(2-\delta)(B-A)\}} \tag{2.3}$$

Consequently, we obtain

$$|f(z)| \geq a_1|z| - \sum_{n=2}^{\infty} a_n |z|^n \geq a_1|z| - |z|^2 \sum_{n=2}^{\infty} a_n \geq a_1\left[|z| - \frac{(2-\delta)(3-\delta)(B-A)|z|^2}{2\{1+B+(2-\delta)(B-A)\}}\right]$$

and

$$|f(z)| \leq a_1|z| + \sum_{n=2}^{\infty} a_n |z|^n \leq a_1|z| + |z|^2 \sum_{n=2}^{\infty} a_n \leq a_1\left[|z| + \frac{(2-\delta)(3-\delta)(B-A)|z|^2}{2\{1+B+(2-\delta)(B-A)\}}\right].$$

which is equivalent to (2.0).

Further, by using second inequality in (2.2), we observe that

$$R = \frac{a_1}{\Gamma(2-\delta)}\left[1 + \frac{(3-\delta)(B-A)}{\{(1+B)+(2-\delta)(B-A)\}}\right] \leq a_1|z| - |z|^2 \sum_{n=2}^{\infty} \phi(\delta,n) a_n$$
$$\leq a_1\left[|z| - \frac{(3-\delta)(B-A)}{\{1+B+(2-\delta)(B-A)\}}|z|^2\right]$$

and

$$\left|\Gamma(2-\delta) z^{\delta} D_z^{\delta} f(z)\right| \leq a_1|z| + \sum_{n=2}^{\infty} \phi(\delta,n) a_n |z|^n \leq a_1|z| + |z|^2 \sum_{n=2}^{\infty} \phi(\delta,n) a_n$$
$$\leq a_1\left[|z| + \frac{(3-\delta)(B-A)}{\{(1+B)+(2-\delta)(B-A)\}}|z|^2\right]$$

which is equivalent to (2.1). Hence the theorem.

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