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The Applications of the Modified Generalized Gamma Distribution in Inventory Control

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Abstract

The main objective of this thesis is the determination of the protection lost sales P_R (i.e., the probability of not going out of stock) and the potential lost sales S_R (i.e., the unsatisfied demand) when the lead time demand has the modified generalized gamma distribution. Using the maximum likelihood method the five unknown parameters of the modified generalized gamma distribution are estimated. The protection and the complement of protection lost sales, the mean and the variance of potential lost sales for the modified generalized gamma distribution and its special cases were estimated. Finally some conclusions are presented.

Keywords: Generalized gamma and incomplete generalized gamma functions; Modified generalized gamma model; Protection lost sales; Mean and the variance of potential lost sales; Lead time demand; The maximum likelihood method

1. Introduction

The main problem in inventory control is the determination of protection lost sales and potential lost sales when both demand and lead time have a certain probability distribution. If the probability density function *(p.d.f)* of demand during lead time is $f(x)$, then for a reorder level system of control with reorder R the protection P_R, the mean of potential lost sales S_R and the variance of potential lost sales V_R are given by Burgin and Wild (1967) as:

$$P_R = \int_0^R f(x)\,dx \qquad , \qquad 0 \le x \le \infty \tag{1-1}$$

$$S_R = \int_R^\infty (x-R)\, f(x)\,dx \qquad , \qquad 0 \le x \le \infty \tag{1-2}$$

$$V_R = \int_R^\infty \left(x - R - S_R\right)^2 f(x)\,dx \tag{1-3}$$

Where, x is random variable that represents the demand for a particular time period, and R is the reorder level.

To determine the other parameters P_R and S_R we must find the distribution of demand during lead time which has the following general characteristics [Burgin and Wild (1975)]:

a- It exists only for non-negative values of demand.

b- As the mean demand of items increases the distribution changes from:

i) Monotonic decreasing to

ii) Unimodal distribution heavily skewed to the right, and finally to

iii) Normally type distribution (truncated at zero).

So, the problem is then to find a distribution which has these general characteristics. Since the previous characteristics are satisfied in the modified generalized gamma distribution, so the main objective of this paper is to determine the protection lost sales P_R and the mean of potential lost sales S_R, when the lead time demand has the modified generalized gamma distribution which was introduced by Shalaby et .al. (2002) with the following p.d.f.

$$f\;(x;\alpha,k,\theta,\lambda,\beta) = \frac{\beta}{\theta\Gamma_\lambda(\alpha,k)} \left(\frac{x}{\theta}\right)^{\alpha\beta\text{-}1} \left[\left(\frac{x}{\theta}\right)^{\beta} + k\right]^{-\lambda} e^{-\left(\frac{x}{\theta}\right)^{\beta}} \tag{1-4}$$

$$, \; x > 0, \quad \lambda \ge 0, \quad \alpha,\beta,k,\theta > 0$$

Where, $\Gamma_\lambda(\alpha,k)$ is the generalized gamma function defined as:

$$\Gamma_\lambda(\alpha,k) = \int_0^\infty y^{\alpha-1}(y+k)^{-\lambda} e^{-y}\,dy \tag{1-5}$$

for a positive integer λ. Here α and k are parameters of the function. This statistical model is mathematically complex because it contains special mathematical functions, such as the gamma function and the new form of generalized gamma function which was presented by Kobayashi (1991). The importance of the modified generalized gamma distribution is that it can be specialized to several important distributions in the field of inventory analysis as: Agarwal and Kalla's model (1996) when $\beta = 1, \theta = 1/b$ and $k = bm$, Agarwal and Al- Saleh's model (2001) for $\beta = 1$ and $\theta = 1/b$, Hoq et. al. model (1974) for $\lambda = 0$ and $\alpha = q/\beta$, Stacy's model (1962) when $\lambda = 0$, the 2-parameter gamma distribution for $\lambda = 0$ and $\beta = 1$, Weibull distribution for $\lambda = 0$ and $\alpha = 1$, Erlang distribution when $\lambda = 0$, $\beta = 1$ and $b = 1/\theta$, exponential distribution when $\lambda = 0$ and $\alpha = \beta = 1$, and finally truncated normal distribution when $\lambda = 0$, $\alpha = 1/2$ and $\beta = 2$.

2. Probability of Stock Out, Mean and Variance of Potential Lost Sales for the Modified Generalized Gamma Distribution

If we operate a classic reorder level system of inventory control then the protection P_R for the modified generalized gamma distribution defined by p.d.f. (1-4) and a reorder level R is:

$$P_R = \int_0^R \frac{\beta}{\theta \Gamma_\lambda(\alpha,k)} \left(\frac{x}{\theta}\right)^{\alpha\beta\text{-}1} \left[\left(\frac{x}{\theta}\right)^{\beta} + k\right]^{-\lambda} e^{-\left(\frac{x}{\theta}\right)^{\beta}} dx$$

$$= \frac{\Gamma_\lambda(\alpha,k,C)}{\Gamma_\lambda(\alpha,k)} = \mathrm{I}_\lambda(\alpha,k,C) \qquad (2-1)$$

where,

$\mathrm{I}_\lambda(\alpha,k,C)$, is the incomplete generalized gamma ratio, with $C = \left(x_\circ/\theta\right)^{\beta}$,

$\Gamma_\lambda(\alpha,k,C)$, is the incomplete generalized gamma function, defined as:

$$\Gamma_\lambda(\alpha,k,C) = \int_0^C y^{\alpha-1}(y+k)^{-\lambda} e^{-y} dy$$

$$= \Gamma_\lambda(\alpha,k) - \gamma_\lambda(\alpha,k,C) \qquad (2\text{-}2)$$

where,

$$\gamma_\lambda(\alpha,k,C) = \int_C^\infty y^{\alpha-1}(y+k)^{-\lambda} e^{-y} dy \qquad (2\text{-}3)$$

The probability of going out of stock H_R is the complement of probability of not going out of stock P_R which is computed as: [Chang, et. al. (2006)]

$$H_R = 1 - \mathrm{I}_\lambda(\alpha,k,C) \qquad (2\text{-}4)$$

On the occasion when a stock out occurs it is desirable to have a measure of the unsatisfied demand (potential lost sales), so for a reorder level R the mean of the potential lost sales S_R of the modified generalized gamma distribution is given by:

$$S_R = \int_R^\infty (x-R) \frac{\beta}{\theta \Gamma_\lambda(\alpha,k)} \left(\frac{x}{\theta}\right)^{\alpha\beta\text{-}1} \left[\left(\frac{x}{\theta}\right)^{\beta} + k\right]^{-\lambda} e^{-\left(\frac{x}{\theta}\right)^{\beta}} dx$$

$$= \frac{1}{\Gamma_\lambda(\alpha,k)} \left[\theta\left[\Gamma_\lambda\left(\alpha+\frac{1}{\beta},k\right) - \Gamma_\lambda\left(\alpha+\frac{1}{\beta},k,C\right)\right] - R\left[\Gamma_\lambda(\alpha,k) - \Gamma_\lambda(\alpha,k,C)\right]\right] \qquad (2\text{-}5)$$

The variance of potential lost sales for the modified generalized gamma distribution is,

$$V_R = \frac{\theta^2}{\Gamma_\lambda(\alpha,k)}\left[\Gamma_\lambda\left(\alpha+\frac{2}{\beta},k\right)-\Gamma_\lambda\left(\alpha+\frac{2}{\beta},k,C\right)\right]-\frac{2(R+S_R)\theta}{\Gamma_\lambda(\alpha,k)}\left[\Gamma_\lambda\left(\alpha+\frac{1}{\beta},k\right)-\Gamma_\lambda\left(\alpha+\frac{1}{\beta},k,C\right)\right]+\frac{2(RS_R+R^2+S_R^2)}{\Gamma_\lambda(\alpha,k)}\left[\Gamma_\lambda(\alpha,k)-\Gamma_\lambda(\alpha,k,C)\right] \quad (2\text{-}6)$$

3. Numerical Illustration

Using the maximum likelihood method, the estimators of five unknown parameters α, κ, θ, λ and β, can be obtained by taking the natural log of the likelihood function of a random sample consisting of observation x_i , $i=1,2,...,n$ from a distribution with p.d.f. (1-3) is [Shalaby et. al. (2002)]:

$$\ell nL = n\,\ell n(\beta) - n\alpha\beta\,\ell n(\theta) - n\,\ell n\Gamma_\lambda(\alpha,k) + (\alpha\beta-1)\sum_{i=1}^{n}\ell n(x_i) - \lambda\sum_{i=n}^{n}\ell n(D_i) - \sum_{i=1}^{n}(V_i^\beta) \quad (3\text{-}1)$$

where, $V_i=(x_i/\theta)$ and $D_i=\left(V_i^\beta+k\right)$

Differentiate ℓnL in (3-1) with respect to each of the five unknown parameters to get,

$$\frac{\partial\,\ell nL}{\partial\,\alpha} = -n\beta\cdot\ell n(\theta) - n\left[\frac{\partial\Gamma_\lambda(\alpha,k)}{\partial\alpha}\Big/\Gamma_\lambda(\alpha,k)\right] + \beta\sum_{i=1}^{n}\ell n(x_i),$$

$$\frac{\partial\,\ell nL}{\partial\,\mathrm{k}} = -n\left[\frac{\partial\Gamma_\lambda(\alpha,k)}{\partial k}\Big/\Gamma_\lambda(\alpha,k)\right] - \lambda\sum_{i=1}^{n}D_i^{-1},$$

$$\frac{\partial\,\ell nL}{\partial\,\theta} = \frac{-n\alpha\beta}{\theta} + \frac{\beta}{\theta}\sum_{i=1}^{n}\left(V_i^\beta\right) + \frac{\lambda\beta}{\theta}\sum_{i=1}^{n}\left(V_i^\beta D_i^{-1}\right), \quad (3\text{-}2)$$

$$\frac{\partial\,\ell nL}{\partial\,\lambda} = -n\left[\frac{\partial\Gamma_\lambda(\alpha,k)}{\partial\lambda}\Big/\Gamma_\lambda(\alpha,k)\right] - \sum_{i=1}^{n}\ell n(D_i),$$

and

$$\frac{\partial\,\ell nL}{\partial\,\beta} = \frac{n}{\beta} - n\alpha\,\ell n(\theta) + \alpha\sum_{i=1}^{n}\ell n(x_i) - \lambda\sum_{i=1}^{n}\left(V_i^\beta D_i^{-1}\,\ell n(V_i)\right) - \sum_{i=1}^{n}\left(V_i^\beta\,\ell n(V_i)\right)$$

From Gradshteyn and Ryzhik, (1980), we have,

$$\frac{\partial\Gamma_\lambda(\alpha,k)}{\partial\alpha} = \int_0^\infty x^{\alpha-1}(x+k)^{-\lambda}e^{-x}\,\ell n(x)\,dx,$$

$$\frac{\partial\Gamma_\lambda(\alpha,k)}{\partial k} = -\lambda\int_0^\infty x^{\alpha-1}(x+k)^{-(\lambda+1)}e^{-x}\,dx = -\lambda\,\Gamma_{(\lambda+1)}(\alpha,k),$$

$$\frac{\partial \Gamma_\lambda(\alpha,k)}{\partial \lambda} = -\int_0^\infty x^{\alpha-1}(x+k)^{-\lambda} e^{-x}\, \ell n(x+k)\, dx$$

Setting the above derivatives in equations (3-2) equal to zero and then, solving these five nonlinear likelihood estimating equations numerically to yield the maximum likelihood estimates.

To illustrate the new results, presented in this paper a numerical example using numerical techniques and computer facilities will be given where the measures, P_R , H_R , S_R and V_R are determined.

A random sample of size (40) observations was generated from a modified generalized gamma distribution, with $\alpha = 4$, $\kappa = 1$, $\theta = 5$, $\lambda = 1$ and $\beta = 2.5$.The observations are arrayed in ascending order as follow:

4.85	5.07	5.22	5.41	5.56	5.81	5.98	6.15	6.21	6.31
6.42	6.54	6.61	6.90	7.09	7.22	7.34	7.41	7.44	7.55
7.64	7.81	7.92	8.11	8.25	8.32	8.45	8.59	8.71	8.90
9.04	9.25	9.48	9.65	10.19	10.35	10.41	10.55	10.82	11.02

The maximum likelihood estimates of the five unknown parameters can be obtained by equating the five nonlinear likelihood estimating equations (3-2) to zero and solving these equations simultaneously, using MATHCAD PROGRAM 2001 to yields the following MLE's, $\hat{\alpha} = 4.001$, $\hat{k} = 0.911$, $\hat{\theta} = 4.955$, and $\hat{\beta} = 2.5$.

Using MLE's estimates, then for a reorder level system of control with reorder R the protection lost sales P_R, the complement of protection lost sales H_R, the mean of potential lost sales S_R, and the variance of potential lost sales V_R can be obtained in table (3-1) as follow,

TABLE: (3-1)

R	P_R	H_R	S_R	V_R	R	P_R	H_R	S_R	V_R
0	0	1	7.766	2.857	8	0.565	0.435	0.569	1.492
1	0	1	6.766	2.857	9	0.769	0.231	0.241	1.372
2	0	1	5.766	2.856	10	0.903	0.097	0.083	0.982
3	0.001	0.999	4.767	2.839	11	0.968	0.032	0.023	0.515
4	0.008	0.992	3.77	2.711	12	0.992	0.008	0.005	0.196
5	0.046	0.954	2.793	2.336	13	0.998	0.002	0.001	0.054
6	0.149	0.851	1.883	1.825	14	1	0	0	0.011
7	0.335	0.665	1.12	1.527	15	1	0	0	0.002

We note that there is an inverse relationship between the reorder point R and both of the complement of protection lost sales H_R, and the mean of potential lost sales S_R. From the statistical viewpoint, the optimal reorder point is obtained when $R = 14$ units, since at this point the protection P_R is equal to one, and

consequently the complement H_R is equal to zero, therefore there are no orders are not met where, the mean S_R is equal to zero.
The results can be displayed graphically for this case, in figure (3-1).

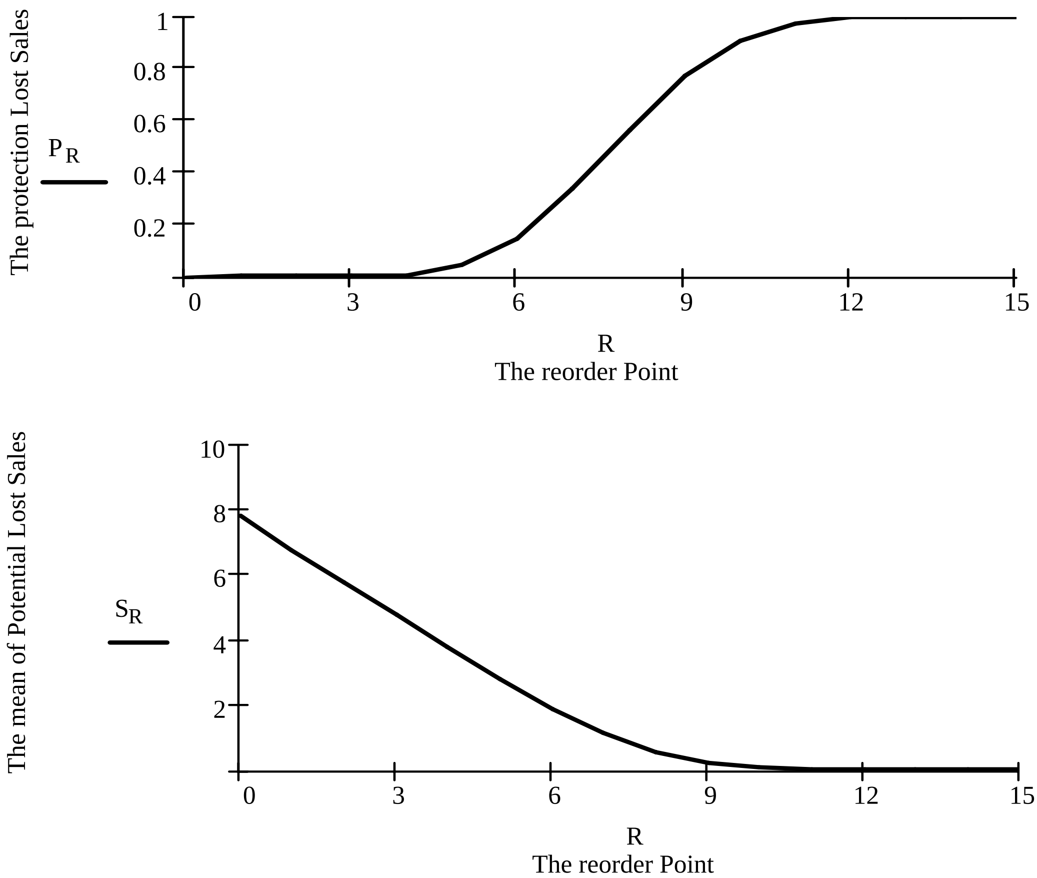

Figure (3-1): The optimal reorder point for the modified generalized gamma distribution.

4. Some Special Cases

The modified generalized gamma model (1-4) can be specialized to many important distributions in the field of inventory analysis as follow:

(4-1) Agarwal and Kalla's model (1996):

When $\beta=1$, $b=1/\theta$ and $m=k/b$ the p.d.f. (1-4) can be specialized to Agarwal and Kall's model (1996), then for $\hat{\alpha}=4.001$, $\hat{k}=0.911$, $\hat{\theta}=4.955$, $\hat{\lambda}=0.872$, $\beta=1$, with reorder level system of control R, the parameters P_R , H_R , S_R and V_R are:

$P_R = \mathrm{I}_{\lambda}(\alpha,bm,C)$,

$H_R = 1-\mathrm{I}_{\lambda}(\alpha,bm,C)$,

$$S_R = \frac{1}{\Gamma_\lambda(\alpha,bm)}\left[\frac{1}{b}\left(\Gamma_\lambda\left(\alpha+1,bm\right)-\Gamma_\lambda\left(\alpha+1,bm,C\right)\right)-R\,\left(\Gamma_\lambda(\alpha,bm)-\Gamma_\lambda(\alpha,bm,C)\right)\right]$$

and

$$V_R = \frac{1}{b^2\,\Gamma_\lambda(\alpha,bm)}\,\left[\Gamma_\lambda\left(\alpha+2,bm\right)-\Gamma_\lambda\left(\alpha+2,bm,C\right)\right]-\frac{2(R+S_R)}{b\Gamma_\lambda(\alpha,bm)}\,\left[\Gamma_\lambda\left(\alpha+1,bm\right)-\right.$$

$$\left.\Gamma_\lambda\left(\alpha+1,bm,C\right)\right]+\frac{2(RS_R+R^2+S_R^2)}{\Gamma_\lambda(\alpha,bm)}\left[\Gamma_\lambda\left(\alpha,bm\right)-\Gamma_\lambda\left(\alpha,bm,C\right)\right] \qquad (4-1)$$

The results can be obtained in table (4-1) as follow,

TABLE: (4-1)

R	P_R	H_R	S_R	V_R	R	P_R	H_R	S_R	V_R
0	0	1	16.598	78.659	29	0.907	0.093	0.633	7.477
1	0	1	15.598	78.613	30	0.919	0.081	0.546	6.491
2	0.002	0.998	14.599	78.129	31	0.93	0.07	0.471	5.621
3	0.009	0.991	13.604	76.784	32	0.939	0.061	0.405	4.856
4	0.023	0.977	12.619	74.471	33	0.947	0.053	0.348	4.186
5	0.044	0.956	11.652	71.35	34	0.954	0.046	0.299	3.601
6	0.073	0.927	10.709	67.697	35	0.96	0.04	0.256	3.093
7	0.108	0.892	9.799	63.791	36	0.966	0.034	0.22	2.651
8	0.15	0.85	8.928	59.85	37	0.971	0.029	0.188	2.27
9	0.196	0.804	8.1	56.01	38	0.975	0.025	0.161	1.94
10	0.245	0.755	7.32	52.337	39	0.978	0.022	0.137	1.656
11	0.296	0.704	6.591	48.844	40	0.981	0.019	0.117	1.412
12	0.348	0.652	5.913	45.517	41	0.984	0.016	0.1	1.203
13	0.4	0.6	5.287	42.329	42	0.986	0.014	0.085	1.023
14	0.451	0.549	4.712	39.255	43	0.988	0.012	0.073	0.87
15	0.5	0.5	4.188	36.278	44	0.99	0.01	0.062	0.739
16	0.546	0.454	3.711	33.389	45	0.991	0.009	0.053	0.627
17	0.591	0.409	3.28	30.592	46	0.993	0.007	0.045	0.531
18	0.632	0.368	2.892	27.894	47	0.994	0.006	0.038	0.45
19	0.671	0.329	2.543	25.307	48	0.995	0.005	0.032	0.381
20	0.706	0.294	2.232	22.846	49	0.995	0.005	0.027	0.322
21	0.739	0.261	1.954	20.523	50	0.996	0.004	0.023	0.273
22	0.768	0.232	1.708	18.348	51	0.997	0.003	0.02	0.23
23	0.795	0.205	1.49	16.329	52	0.997	0.003	0.017	0.194
24	0.819	0.181	1.297	14.47	53	0.998	0.002	0.014	0.164
25	0.841	0.159	1.127	12.77	54	0.998	0.002	0.012	0.138
26	0.86	0.14	0.978	11.227	55	0.998	0.002	0.01	0.117
27	0.878	0.122	0.847	9.836	56	0.999	0.001	0.008	0.098
28	0.893	0.107	0.733	8.589	57	0.999	0.001	0.007	0.083
R	P_R	H_R	S_R	V_R	R	P_R	H_R	S_R	V_R
58	0.999	0.001	0.006	0.07	61	0.999	0.001	0.004	0.042
59	0.999	0.001	0.005	0.059	62	0.999	0.001	0.003	0.035
60	0.999	0.001	0.004	0.049	63	1	0	0.003	0.029
					64	1	0	0.002	0.025

For this case the optimal reorder point is given when $R = 63$ units.

(4-2) Agarwal and Al-Saleh's model (2001):

For $\beta = 1$ and $b = 1/\theta$ the p.d.f (1-4) introduces Agarwal and AL-Saleh's model (2001), then the maximum likelihood estimates are:

$\hat{\alpha} = 4.001$, $\hat{k} = 0.911$, $\hat{\theta} = 4.955$, $\hat{\lambda} = 0.872$, $\beta = 1$, then for a reorder level system of control R, the parameters P_R , H_R , S_R and V_R are obtained as:

$P_R = \mathrm{I}_\lambda(\alpha, k, C)$,

$H_R = 1 - \mathrm{I}_\lambda(\alpha, k, C)$,

$$S_R = \frac{1}{\Gamma_\lambda(\alpha,k)} \left[\frac{1}{b}\left(\Gamma_\lambda(\alpha+2,k) - \Gamma_\lambda(\alpha+2,k,C)\right) - R\left(\Gamma_\lambda(\alpha,k) - \Gamma_\lambda(\alpha,k,C)\right) \right] \quad \text{and}$$

$$V_R = \frac{1}{b^2\Gamma_\lambda(\alpha,k)} \left[\Gamma_\lambda(\alpha+2,k) - \Gamma_\lambda(\alpha+2,k,C)\right] - \frac{2(R+S_R)}{b\Gamma_\lambda(\alpha,k)} \left[\Gamma_\lambda(\alpha+1,k) - \Gamma_\lambda(\alpha+1,k,C)\right] + \frac{2(R S_R + R^2 + S_R^2)}{\Gamma_\lambda(\alpha,k)} \left[\Gamma_\lambda(\alpha,k) - \Gamma_\lambda(\alpha,k,C)\right] \qquad (4-2)$$

the results can be obtained in table (4-2).

TABLE: (4-2)

R	P_R	H_R	S_R	V_R	R	P_R	H_R	S_R	V_R
0	0	1	16.598	78.659	16	0.546	0.454	3.711	33.389
1	0	1	15.598	78.613	17	0.591	0.409	3.28	30.592
2	0.002	0.998	14.599	78.129	18	0.632	0.368	2.892	27.894
3	0.009	0.991	13.604	76.784	19	0.671	0.329	2.543	25.307
4	0.023	0.977	12.619	74.471	20	0.706	0.294	2.232	22.846
5	0.044	0.956	11.652	71.35	21	0.739	0.261	1.954	20.523
6	0.073	0.927	10.709	67.697	22	0.768	0.232	1.708	18.348
7	0.108	0.892	9.799	63.791	23	0.795	0.205	1.49	16.329
8	0.15	0.85	8.928	59.85	24	0.819	0.181	1.297	14.47
9	0.196	0.804	8.1	56.01	25	0.841	0.159	1.127	12.77
10	0.245	0.755	7.32	52.337	26	0.86	0.14	0.978	11.227
11	0.296	0.704	6.591	48.844	27	0.878	0.122	0.847	9.836
12	0.348	0.652	5.913	45.517	28	0.893	0.107	0.733	8.589
13	0.4	0.6	5.287	42.329	29	0.907	0.093	0.633	7.477
14	0.451	0.549	4.712	39.255	30	0.919	0.081	0.546	6.491
15	0.5	0.5	4.188	36.278	31	0.93	0.07	0.471	5.621

R	P_R	H_R	S_R	V_R	R	P_R	H_R	S_R	V_R
32	0.939	0.061	0.405	4.856	48	0.995	0.005	0.032	0.381
33	0.947	0.053	0.348	4.186	49	0.995	0.005	0.027	0.322
34	0.954	0.046	0.299	3.601	50	0.996	0.004	0.023	0.273
35	0.96	0.04	0.256	3.093	51	0.997	0.003	0.02	0.23
36	0.966	0.034	0.22	2.651	52	0.997	0.003	0.017	0.194
37	0.971	0.029	0.188	2.27	53	0.998	0.002	0.014	0.164
38	0.975	0.025	0.161	1.94	54	0.998	0.002	0.012	0.138
39	0.978	0.022	0.137	1.656	55	0.998	0.002	0.01	0.117
40	0.981	0.019	0.117	1.412	56	0.999	0.001	0.008	0.098
41	0.984	0.016	0.1	1.203	57	0.999	0.001	0.007	0.083
42	0.986	0.014	0.085	1.023	58	0.999	0.001	0.006	0.07
43	0.988	0.012	0.073	0.87	59	0.999	0.001	0.005	0.059
44	0.99	0.01	0.062	0.739	60	0.999	0.001	0.004	0.049
45	0.991	0.009	0.053	0.627	61	0.999	0.001	0.004	0.042
46	0.993	0.007	0.045	0.531	62	0.999	0.001	0.003	0.035
47	0.994	0.006	0.038	0.45	63	1	0	0.003	0.029
					64	1	0	0.002	0.025

The optimal reorder point for Agarwal and Al-Saleh's model (2001) is obtained when $R = 63$ units.

(4-3) Hoq, Ali, and Templetion's model (1974):

For $\lambda = 0$ and $\alpha = q/\beta$ the p.d.f (1-4) reduce Hoq et. al. model (1974), then for a reorder level system of control R, the parameters P_R , H_R , S_R and V_R are obtained as follow:

$$P_R = \mathrm{I}\left(q/\beta,\, C\right),$$

$$H_R = 1 - \mathrm{I}\left(q/\beta,\, C\right),$$

$$S_R = \frac{1}{\Gamma(q/\beta)}\left[\theta\left(\Gamma\left(q+1/\beta\right) - \Gamma\left(q+1/\beta, C\right)\right) - R\left(\Gamma\left(q/\beta\right) - \Gamma\left(\left(q/\beta\right), C\right)\right)\right]$$

and

$$V_R = \frac{\theta^2}{\Gamma(q/\beta)}\left[\Gamma\left(q+2/\beta\right) - \Gamma\left(q+2/\beta, C\right)\right] - \frac{2(R+S_R)\theta}{\Gamma(q/\beta)}\left[\Gamma\left(q+1/\beta\right) - \Gamma\left(q+1/\beta, C\right)\right] + \frac{2(R S_R + R^2 + S_R^2)}{\Gamma(q/\beta)}\left[\Gamma\left(q/\beta\right) - \Gamma\left(\left(q/\beta\right), C\right)\right] \qquad (4-3)$$

The results are shown in table (4-3).

TABLE: (4-3)

R	P_R	H_R	S_R	V_R	R	P_R	H_R	S_R	V_R
0	0	1	8.372	2.923	8	0.422	0.578	0.883	0.98
1	0	1	7.372	2.923	9	0.648	0.352	0.42	0.534
2	0	1	6.372	2.923	10	0.829	0.171	0.164	0.221
3	0	1	5.372	2.916	11	0.934	0.066	0.052	0.067
4	0.003	0.997	4.373	2.853	12	0.981	0.019	0.013	0.015
5	0.02	0.98	3.382	2.607	13	0.996	0.004	0.002	0.003
6	0.081	0.919	2.428	2.111	14	0.999	0.001	0	0
7	0.215	0.785	1.569	1.515	15	1	0	0	0
					16	1	0	0	0

For the case of Hoq et al. model (1974), the optimal reorder point is given in table (4-3) when $R = 15$ units.

(4-4) Stacy's model (1962):

When $\lambda = 0$, the p.d.f (1-4) gives the Stacy's model (1962). By substituting with the estimated parameters $\hat{\alpha}, \hat{k}, \hat{\theta}, \hat{\lambda}$ and $\hat{\beta}$ in p.d.f. (1-4) the parameters P_R, H_R, S_R and V_R can be calculated based on the last p.d.f. as follow,

$P_R = \mathrm{I}(\alpha, C)$,

$H_R = 1 - \mathrm{I}(\alpha, C)$,

$$S_R = \frac{1}{\Gamma(\alpha)}\left[\theta\left(\Gamma\left(\alpha + 1/\beta\right) - \Gamma\left(\alpha + 1/\beta, C\right)\right) - R\left(\Gamma(\alpha) - \Gamma(\alpha, C)\right)\right] \quad \text{and}$$

$$V_R = \frac{\theta^2}{\Gamma(\alpha)}\left[\Gamma\left(\alpha + 2/\beta\right) - \Gamma\left(\alpha + 2/\beta, C\right)\right] - \frac{2(R + S_R)\theta}{\Gamma(\alpha)}\left[\Gamma\left(\alpha + 1/\beta\right) - \Gamma\left(\alpha + 1/\beta, C\right)\right] + \frac{2(R S_R + R^2 + S_R^2)}{\Gamma(\alpha)}\left[\Gamma(\alpha) - \Gamma(\alpha, C)\right] \quad (4-4)$$

The results of this case are shown in table (4-4).

TABLE: (4-4)

R	P_R	H_R	S_R	V_R	R	P_R	H_R	S_R	V_R
0	0	1	8.372	2.923	9	0.648	0.352	0.42	0.534
1	0	1	7.372	2.923	10	0.829	0.171	0.164	0.221
2	0	1	6.372	2.923	11	0.934	0.066	0.052	0.067
3	0	1	5.372	2.916	12	0.981	0.019	0.013	0.015
4	0.003	0.997	4.373	2.853	13	0.996	0.004	0.002	0.003
5	0.02	0.98	3.382	2.607	14	0.999	0.001	0	0
6	0.081	0.919	2.428	2.111	15	1	0	0	0
7	0.215	0.785	1.569	1.515	16	1	0	0	0
8	0.422	0.578	0.883	0.98					

For the case of Stacy's model, the optimal reorder point is given in table (4-4) when $R = 15$ units.

(4-5) The 2- parameter Weibull distribution:

When $\lambda = 0$ and $\alpha = 1$ the p.d.f (1-4) can be specialized to 2-parameter Weibull distribution, which was studied by Tadikamala (1978). Then for

$\alpha = 1$, $k = 0.911$, $\theta = 4.955$, $\lambda = 0$ and $\beta = 2.5$ the p.d.f.(1-4) with a reorder level system of control R , the parameters P_R , H_R , S_R , and V_R become,

$P_R = \mathrm{I}(1, C)$,

$H_R = 1 - \mathrm{I}(1, C)$,

$S_R = \left[\theta\left(\Gamma\left(1 + 1/\beta\right) - \Gamma\left(1 + 1/\beta, C\right)\right) - R\left(1 - \Gamma(1, C)\right) \right]$ and

$$V_R = \theta^2 \left[\Gamma\left(1 + 2/\beta\right) - \Gamma\left(1 + 2/\beta, C\right)\right] - 2\theta(R + S_R)\left[\Gamma\left(1 + 1/\beta\right) - \Gamma\left(1 + 1/\beta, C\right)\right] + 2(R S_R + R^2 + S_R^2)\left[1 - \Gamma\left(1, C\right)\right] \quad (4-5)$$

The results are shown in table (4-5) as follow:

TABLE: (4-5)

R	P_R	H_R	S_R	V_R	R	P_R	H_R	S_R	V_R
0	0	1	4.396	3.539	7	0.907	0.093	0.092	0.144
1	0.018	0.982	3.402	3.292	8	0.964	0.036	0.031	0.045
2	0.098	0.902	2.454	2.617	9	0.988	0.012	0.009	0.011
3	0.248	0.752	1.622	1.897	10	0.997	0.003	0.002	0.002
4	0.443	0.557	0.965	1.28	11	0.999	0.001	0	0
5	0.64	0.36	0.509	0.759	12	1	0	0	0
6	0.801	0.199	0.234	0.37	13	1	0	0	0

For the case of 2-parameter Weibull distribution, the optimal reorder point is given in table (4-5) when $R = 12$ units.

(4-6) The 2- parameter gamma distribution:

For $\lambda = 0$ and $\beta = 1$ the p.d.f (1-4) introduces the 2-parameter gamma distribution, which was discussed by Burgin (1975). Using the same values for the estimated parameters then the values of P_R, H_R, S_R and V_R can be calculated as follow,

$P_R = \mathrm{I}(\alpha, C)$,

$H_R = 1 - \mathrm{I}(\alpha, C)$,

$S_R = \frac{1}{\Gamma(\alpha)} \left[\theta\left(\Gamma\left(\alpha + 1\right) - \Gamma\left(\alpha + 1, C\right)\right) - R\left(\Gamma(\alpha) - \Gamma(\alpha, C)\right) \right]$ and

$$V_R = \frac{\theta^2}{\Gamma(\alpha)}\left[\Gamma\left(\alpha + 2\right) - \Gamma\left(\alpha + 2, C\right)\right] - \frac{2(R + S_R)\theta}{\Gamma(\alpha)}\left[\Gamma\left(\alpha + 1\right) - \Gamma\left(\alpha + 1, C\right)\right] + \frac{2(R S_R + R^2 + S_R^2)}{\Gamma(\alpha)}\left[\Gamma\left(\alpha\right) - \Gamma\left(\alpha, C\right)\right] \quad (4-6)$$

The results of this case are shown in table (4-6).

TABLE: (4-6)

R	P_R	H_R	S_R	V_R	R	P_R	H_R	S_R	V_R
0	0	1	19.825	98.233	36	0.931	0.069	0.494	6.29
1	0	1	18.825	98.211	37	0.94	0.06	0.43	5.482
2	0.001	0.999	17.825	97.966	38	0.947	0.053	0.373	4.767
3	0.003	0.997	16.827	97.175	39	0.954	0.046	0.323	4.138
4	0.009	0.991	15.833	95.614	40	0.96	0.04	0.28	3.586
5	0.02	0.98	14.847	93.225	41	0.965	0.035	0.242	3.103
6	0.035	0.965	13.874	90.09	42	0.969	0.031	0.21	2.681
7	0.055	0.945	12.918	86.381	43	0.973	0.027	0.181	2.313
8	0.081	0.919	11.986	82.301	44	0.977	0.023	0.156	1.992
9	0.111	0.889	11.082	78.04	45	0.98	0.02	0.135	1.715
10	0.146	0.854	10.21	73.751	46	0.983	0.017	0.116	1.474
11	0.184	0.816	9.375	69.538	47	0.985	0.015	0.1	1.265
12	0.226	0.774	8.58	65.466	48	0.987	0.013	0.086	1.085
13	0.269	0.731	7.827	61.554	49	0.989	0.011	0.074	0.93
14	0.314	0.686	7.118	57.801	50	0.99	0.01	0.063	0.796
15	0.359	0.641	6.454	54.192	51	0.992	0.008	0.054	0.681
16	0.404	0.596	5.835	50.707	52	0.993	0.007	0.047	0.582
17	0.448	0.552	5.261	47.328	53	0.994	0.006	0.04	0.497
18	0.492	0.508	4.731	44.043	54	0.995	0.005	0.034	0.424
19	0.533	0.467	4.244	40.847	55	0.995	0.005	0.029	0.362
20	0.573	0.427	3.797	37.74	56	0.996	0.004	0.025	0.309
21	0.611	0.389	3.39	34.729	57	0.997	0.003	0.021	0.263
22	0.647	0.353	3.02	31.825	58	0.997	0.003	0.018	0.224
23	0.681	0.319	2.684	29.039	59	0.998	0.002	0.015	0.19
24	0.712	0.288	2.381	26.384	60	0.998	0.002	0.013	0.162
25	0.741	0.259	2.107	23.872	61	0.998	0.002	0.011	0.138
26	0.768	0.232	1.862	21.51	62	0.998	0.002	0.01	0.117
27	0.792	0.208	1.642	19.305	63	0.999	0.001	0.008	0.099
28	0.815	0.185	1.446	17.26	64	0.999	0.001	0.007	0.084
29	0.835	0.165	1.271	15.377	65	0.999	0.001	0.006	0.071
30	0.853	0.147	1.115	13.652	66	0.999	0.001	0.005	0.06
31	0.87	0.13	0.977	12.081	67	0.999	0.001	0.004	0.051
32	0.885	0.115	0.855	10.659	68	0.999	0.001	0.004	0.043
33	0.899	0.101	0.747	9.378	69	0.999	0.001	0.003	0.037
34	0.911	0.089	0.652	8.229	70	1	0	0.003	0.031
35	0.921	0.079	0.568	7.203	71	1	0	0.002	0.026

The optimal reorder point for the 2-parameter gamma distribution is obtained when $R = 70$ units.

5. Summary and Conclusions

From the obtained results in this paper we note the following points.

1. In general there is an inverse relationship between the reorder level R and both of the complement of protection lost sales H_R, and the mean of potential lost sales S_R.
2. From the statistical viewpoint, the optimal reorder point R will be achieved when the protection lost sales P_R is equal to one, and consequently the complement of protection lost sales H_R is equal to zero, therefore there are no orders are not met where, the mean of potential lost sales S_R is equal to zero.
3. From all previous cases the best results can be obtained when the optimal reorder point R is the least value, accordingly Weibull distribution is the best model where $R = 12$. This conclusion is consistent with the findings of Tadikamala (1978).
4. From the statistical viewpoint, the more the number of model parameters as the results are better.
5. It is noted that the models that contain generalized gamma function are better than these models that contain ordinary gamma function.
6. We can estimate the parameters P_R, H_R, S_R and V_R for other distributions such as; Erlang distribution, exponential distribution and truncated normal distribution as special cases of the modified generalized gamma distribution.
7. All computations, in the thesis are performed using computer facilities and MATHCAD PROGRAM (2001).
8. Finally, all previous conclusions represent pure statistical viewpoint.

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