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A Coincidence Point Theorem for Four Mappings in Dislocated Metric Spaces

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Abstract

In this paper, we prove a common coincidence point theorem for four mappings in dislocated metric spaces.

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1 Introduction

In 2005 F.M. Zeyada, G.H. Hassan, M.A. Ahmed [3] established various definitions of dislocated quasi-metric space. C.T. Aage, J.N. Salunke [2] and A.Isufati [1] proved fixed point theorems for a single map and a pair of mappings in dislocated metric spaces. In this paper we prove a common coincidence point theorem for four maps in dislocated metric spaces.

Now we give some known definitions in dq-metric spaces.

Definition 1.1 ([3]).*Let X be a nonempty set and let $d : X \times X \longrightarrow [0, \infty)$ be a function satisfying following conditions : (i) $d(x, y) = d(y, x) = 0$ implies $x = y$, (ii) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.*
Then d is called a dislocated quasi-metric on X.

If d satisfies $d(x,x) = 0, \forall x \in X$ then the dislocated quasi-metric is called a quasi-metric on X.
If d satisfies $d(x,y) = d(y,x), \forall x, y \in X$ then the dislocated quasi-metric is called a dislocated metric on X.

Definition 1.2 ([3]). *A sequence $\{x_n\}$ in dq-metric space (dislocated quasi-metric space) (X,d) is called a Cauchy sequence if for given $\epsilon > 0$,there exists $n_0 \in N$ such that for all $m, n \geq n_0$ implies $d(x_m, x_n) < \epsilon$ or $d(x_n, x_m) < \epsilon$.*

Definition 1.3 ([3]). *A sequence $\{x_n\}$ is said to be dislocated quasi-converges to x if $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0$. In this case x is called a dq-limit of $\{x_n\}$ and we write $x_n \to x$.*

Lemma 1.4 ([3]).*dq-limit in a dq-metric space is unique.*

Definition 1.5 ([3]). *A dq-metric space (X,d) is called complete if every Cauchy sequence in it is dq-convergent.*

Definition 1.6 ([3]). *Let (X, d_1) and (Y, d_2) be dq-metric spaces and let $f : X \longrightarrow Y$ be a function. Then f is said to be continuous at $x_0 \in X$, if the sequence $\{f(x_n)\}$ is d_2q- convergent to $f(x_0) \in Y$ whenever the sequence $\{x_n\}$ in X is d_1q- convergent to x_0.*

Definition 1.7 ([3]). *Let (X,d) be a dq-metric space. A map $T : X \longrightarrow X$ is called contraction if there exists $0 \leq \lambda < 1$ such that $d(Tx, Ty) \leq \lambda d(x, y)$ for all $x, y \in X$.*

Theorem 1.8 ([3]). *Let (X,d) be a dq-metric space and let $T : X \longrightarrow X$ be a continuous contraction mapping. Then T has unique fixed point.*

2 Main Results

Theorem 2.1 *Let (X,d) be a complete dislocated metric space. Let F, G, S and $T : X \to X$ be continuous mappings satisfying*
(2.1.1) $S(X) \subseteq G(X)$ and $T(X) \subseteq F(X)$,
(2.1.2) $SF = FS$ and $TG = GT$ and
(2.1.3) $d(Sx, Ty) \leq \phi(max\{d(Fx, Gy), d(Fx, Sx), d(Gy, Ty), \frac{d(Fx,Sx)d(Gy,Ty)}{d(Fx,Gy)}\})$
for all $x, y \in X$, where $\phi : R^+ \to R^+$ is monotonically non-decreasing and $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for all $t > 0$.
Then (i) F and S (or) G and T have coincidence point or
(ii) the pairs (F,S) and (G,T) have a common coincidence point.

Proof. It is clear that $\phi^n(t) \to 0$ as $n \to \infty$ and $\phi(t) < t$ for all $t > 0$. Suppose x_0 is an arbitrary point of X and define the sequences $\{x_n\}$ and $\{y_n\}$ in X such that
$y_{2n} = Sx_{2n} = Gx_{2n+1}$,
$y_{2n+1} = Tx_{2n+1} = Fx_{2n+2}$, $n = 0, 1, 2, 3, ...$
If $y_{2n} = y_{2n+1}$ for some n, then $Gx_{2n+1} = Tx_{2n+1}$. Hence x_{2n+1} is a coincidence point of G and T. If $y_{2n+1} = y_{2n+2}$ for some n, then $Fx_{2n+2} = Sx_{2n+2}$. Hence x_{2n+2} is a coincidence point of F and S.
Assume that $y_n \neq y_{n+1}$ for all n.

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\leq \phi\left(\max\left\{\begin{array}{c} d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), \\ d(y_{2n}, y_{2n+1}), \frac{d(y_{2n-1},y_{2n})d(y_{2n},y_{2n+1})}{d(y_{2n-1},y_{2n})} \end{array}\right\}\right) \\ &= \phi\left(\max\left\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\right\}\right) \end{aligned}$$

If $max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\} = d(y_{2n}, y_{2n+1})$,then
$d(y_{2n}, y_{2n+1}) \leq \phi(d(y_{2n}, y_{2n+1})) < d(y_{2n}, y_{2n+1})$.It is a contradiction.
Hence $d(y_{2n}, y_{2n+1}) \leq \phi(d(y_{2n-1}, y_{2n}))$ .....(1)

$$\begin{aligned} d(y_{2n-1}, y_{2n}) &= d(Sx_{2n}, Tx_{2n-1}) \\ &\leq \phi\left(\max\left\{\begin{array}{c} d(y_{2n-1}, y_{2n-2}), d(y_{2n-1}, y_{2n}), \\ d(y_{2n-2}, y_{2n-1}), \frac{d(y_{2n-1},y_{2n})d(y_{2n-2},y_{2n-1})}{d(y_{2n-1},y_{2n-2})} \end{array}\right\}\right) \\ &= \phi\left(\max\left\{d(y_{2n-1}, y_{2n-2}), d(y_{2n-1}, y_{2n})\right\}\right) \end{aligned}$$

Thus $d(y_{2n-1}, y_{2n}) \leq \phi(d(y_{2n-2}, y_{2n-1}))$ ..... (2)
From (1) and (2) we have

$$\begin{aligned} d(y_n, y_{n+1}) &\leq \phi\left(d(y_{n-1}, y_n)\right) \\ &\leq \phi^2(d(y_{n-2}, y_{n-1})) \\ &. \\ &. \\ &. \\ &\leq \phi^n\left(d(y_0, y_1)\right) ....(3) \end{aligned}$$

Now for $n, m \in N$ with $n < m$, we have

$$\begin{aligned} d(y_n, y_m) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + ... + d(y_{m-1}, y_m) \\ &\leq \phi^n(d(y_0, y_1)) + \phi^{n+1}(d(y_0, y_1)) + ... + \phi^{m-1}(d(y_0, y_1)) \ from(3) \\ &\leq \sum_{i=n}^{\infty} \phi^i(d(y_0, y_1)) \\ &\to 0 \ as \ n, m \to \infty. \end{aligned}$$

Hence $\{y_n\}$ is a Cauchy sequence in the complete dislocated metric space X. Hence there exists $u \in X$ such that $\{y_n\}$ converges to u.

Since $SF = FS$ and S and F are continuous, we have
$Su = \lim_{n\to\infty} SFx_{2n} = \lim_{n\to\infty} FSx_{2n} = Fu.$
Since $TG = GT$ and T and G are continuous, we have
$Tu = \lim_{n\to\infty} TGx_{2n+1} = \lim_{n\to\infty} GTx_{2n+1} = Gu.$
Thus the pairs (F, S) and (G, T) have a common coincidence point.
The following Example illustrates Theorem 2.1.

Example 2.2 *Let* $X = [0, 1]$ *and* $d(x, y) = max\{x, y\}$*.Then* (X, d) *is a dislocated metric space. Define*
$Sx = 0$*,*$Tx = \frac{x}{6}$*,* $Fx = x$ *and* $Gx = \frac{x}{3}$*.*
Clearly S, T, F, G *are continuous and* (2.1.1) *and* (2.1.2) *are satisfied. Also*
$d(Sx, Ty) \le \phi(d(Fx, Gy))$ *for all* $x, y \in X$*, where* $\phi(t) = \frac{t}{2}$*.*
Clearly 0 *is a common coincidence point of the pairs* (F, S) *and* (G, T)*.*

Recently,Isufati[1] proved the following

Theorem 2.3 (*Theorem* 3.3, [1])*.Let* S, T *be continuous self mappings on a complete dislocated metric space* (X, d) *satisfying*
$d(Sx, Ty) \le h\ max\{d(x, y), d(x, Sx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Sx)]\}$
for all $x, y \in X$ *and* $0 < h < 1$*.*
Then S *and* T *have a common fixed point in* X*.*

We observed that Theorem 2.3 is not valid in view of the following example.

Example 2.4 *Let* $X = \{1, 2, 3\}$ *and* $d(x, y) = \begin{cases} 2, if\ x + y\ is\ even \\ 1, if\ x + y\ is\ odd \end{cases}$.
Define $S, T : X \to X$ *as* $S1 = S2 = S3 = 1$ *and* $T1 = T2 = T3 = 2$*.*
Clearly $d(Sx, Ty) = \frac{1}{2}max\{d(x, y), d(x, Sx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Sx)]\}$
for all $x, y \in X$ *.Clearly* 1 *is a fixed point of* S *and* 2 *is a fixed point of* T*.Also* S *and* T *have no common fixed points.*

Now, we give a corollary which is a probable modification of Theorem 2.3.

Corollary 2.5 *Let* (X, d) *be a complete dislocated metric space and* $S, T : X \to X$ *be continuous mappings satisfying*
(2.5.1) $d(Sx, Ty) \le \phi(max\{d(x, y), d(x, Sx), d(y, Sy), \frac{d(x,Sx)d(y,Ty)}{d(x,y)}\})$
for all $x, y \in X$*, where* ϕ *is as in Theorem* 2.1*.*
Then either S *or* T *has a fixed point or* S *and* T *have a unique common fixed point.*

Proof. Putting F = G = I(Identity map) in Theorem 2.1,we have either S or T has a fixed point or S and T have a common fixed point in X.
Suppose u and v are two common fixed points of S and T. Then
$d(u, v) = d(Su, Tv) \le \phi(max\{d(u, v), d(u, u), d(v, v), \frac{d(u,u)d(v,v)}{d(u,v)}\})$ ... (1)
Replacing v by u in (1), we get

$d(u,u) \leq \phi(d(u,u))$ which implies that $d(u,u) = 0$.
Similarly, replacing u by v in (1), we get
$d(v,v) \leq \phi(d(v,v))$ which implies that $d(v,v) = 0$.
Now from (1),
$d(u,v) \leq \phi(d(u,v)) < d(u,v)$ if $d(u,v) > 0$. Hence $d(u,v) = 0$.
Since (X,d) is a dislocated metric space, we have $u = v$.
Hence S and T have a unique common fixed point.

Theorem 2.6 *Let (X,d) be a complete dq- metric space. Let $T : X \to X$ be a continuous mapping satisfying*
(2.6.1) $d(Tx,Ty) \leq \phi(max\{d(x,y), d(x,Tx)d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{d(x,y)}\})$
for all $x, y \in X$, where ϕ is as in Theorem 2.1.
Then T has a unique common fixed point in X.

Proof. It is clear that $\phi(t) < t$ for all $t > 0$. Suppose x_0 is an arbitrary point of X.
Define $x_{n+1} = Tx_n$, $n = 0, 1, 2, ...$
If $x_{n+1} = x_n$ for some n, then $Tx_n = x_n$. Hence x_n is a fixed point of T.
Assume that $x_{n+1} \neq x_n, \quad \forall n$.

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \phi\left(\max\left\{\begin{array}{c} d(x_{n-1}, x_n), d(x_{n-1}, x_n), \\ d(x_n, x_{n+1}), \frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{d(x_{n-1},x_n)}\}) \end{array}\right\}\right) \\ &= \phi\left(\max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\right\}\right) \end{aligned}$$

If $max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$ then
$d(x_n, x_{n+1}) \leq \phi(d(x_n, x_{n+1})) < d(x_n, x_{n+1})$.
It is a contradiction. Hence

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \phi\left(d(x_{n-1}, x_n)\right) \\ &\leq \phi^2(d(x_{n-2}, x_{n-1})) \\ &. \\ &. \\ &. \\ &\leq \phi^n(d(x_0, x_1)) \end{aligned}$$

As in Theorem 2.1, it follows that $\{x_n\}$ is a Cauchy sequence in complete dq-metric space X. Hence there exists $u \in X$ such that $\{x_n\}$ converges to u.
Since T is continuous, we have $u = \lim_{n\to\infty} Tx_n = T(\lim_{n\to\infty} x_n) = Tu$.
Suppose v is another fixed point of T. Now

$$\begin{aligned} d(u,v) &= d(Tu, Tv) \\ &\leq \phi\left(\max\left\{d(u,v), d(u,u), d(v,v), \frac{d(u,u)d(v,v)}{d(u,v)}\right\}\right) .....(1) \end{aligned}$$

Replacing v by u in (1), we get
$d(u,u) \leq \phi(d(u,u)) < d(u,u)$ if $d(u,u) > 0$.
Hence $d(u,u) = 0$.... (2)
Similarly we can show that $d(v,v) = 0$ .... (3)
Now from(1),$d(u,v) \leq \phi(d(u,v))$ which implies that $d(u,v) = 0$......(4)

$$\begin{aligned} d(v,u) &= d(Tv,Tu) \\ &\leq \phi\left(\max\left\{d(v,u), d(v,v), d(u,u), \tfrac{d(v,v)d(u,u)}{d(v,u)}\right\}\right) \end{aligned}$$

Using(2) and (3),we have
$d(v,u) \leq \phi(d(v,u))$ which implies that $d(v,u) = 0$.......(5)
From (4) and (5), we have $u = v$.
Thus u is the unique fixed point of T.

Remark 2.7 *Theorem* 3.3 *of C.T. Aage, and J.N. Salunke* [2] *is a corollary to Theorem* 2.6*. Theorem* 3.1 *of A. Isufati* [1] *is a corollary to Theorem* 2.6*.*

References

[1] A.Isufati, Fixed Point Theorems in Dislocated Quasi-Metric Space, *Applied Mathematical Sciences*, Vol. **4**, 2010, no. 5, 217 - 223.

[2] C.T. Aage, J.N. Salunke, The Results on Fixed Point in Dislocated and Dislocated Quasi-Metric Space, *Applied Mathematical Scineces*, Vol. **2**, 2008, no. 59, 2941-2948.

[3] F.M. Zeyada, G.H. Hassan, M.A. Ahmed, A generalization of a fixed point theorem due to Hitzler and Seda in dislocated quasi-metric spaces, *The Arabian Journal for Science and Engineering*, Vol.**31**, Number iA, (2005).

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