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Estimation for the Parameters of the Weibull Extension Model Based on Generalized Order Statistics

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Abstract

The estimation problem for the unknown parameters of Weibull extension model (WEM) is investigated based on generalized order statistics. Maximum likelihood estimators and their asymptotic variance-covariance matrix are derived, also Bayes estimators using symmetric squared error loss function are obtained. Some numerical results using simulation study are reported.

Keywords: Weibull extension model; Generalized order statistics; Maximum likelihood estimation; Bayes estimators; Squared error loss function.

1. Introduction

Xie et al. (2002) proposed Weibull extension model, this model has bathtub shaped failure rate function and asymptotically related to the traditional Weibull distribution with two parameter. The new model also is an extension of two parameter model proposed by Chen (2000) which can be used to model bathtub shaped failure rate. The probability density function of WEM is

$$f(x) = \lambda\beta(x/\alpha)^{\beta-1}\exp[(x/\alpha)^{\beta} + \lambda\alpha(1 - e^{(x/\alpha)^{\beta}})], \qquad (1.1)$$

$$\alpha, \lambda, \beta > 0, \qquad x \ge 0$$

the distribution, the reliability and the failure rate functions are given by respectively

$$F(x) = 1 - \exp[-\lambda\alpha(e^{(x/\alpha)^{\beta}} - 1)] \;, \qquad (1.2)$$

$$R(x) = 1 - F(x) = \exp[\lambda\alpha(1 - e^{(x/\alpha)^{\beta}})] \;, \qquad (1.3)$$

$$r(x) = \frac{f(x)}{1 - F(x)} = \lambda\beta(x/\alpha)^{\beta-1}\exp[(x/\alpha)^{\beta}] \;, \qquad (1.4)$$

Xie et al. (2002), used maximum likelihood method for estimating the parameters of the new model under type II censoring. Tang et al. (2003), discussed the proprieties of Weibull extension model, they obtained the maximum likelihood estimators under complete sample scheme. Elshahat (2007-a,b,c), used likelihood function and two sets of quasi-likelihood function to derive Bayesian estimators for the unknown parameters of the Weibull extension model, he studied the estimation problem of the unknown parameters of the Weibull extension model based on a progressively type-II censored sample, also he obtained the estimators of the unknown parameters of the Weibull extension model using maximum quasi-likelihood method.

The concept of generalized order statistics (gOS's) has been introduced by Kamps (1995). Its enable a unified approach to order random variables as ordinary order statistics, sequential order statistics, order statistics with non integral sample size, record values and progressively type II censored order statistics. Let $n \in \mathbb{N}$, $k \ge 1$, $m \in \mathbb{R}$, and $1 \le r \le n-1$ (where *n* is the number of observations and *r* is the number of failures), be parameters such that $\gamma_r = k + (n-r)(m+1)$, then the random variable $X(1,n,m,k),\ldots,X(r,n,m,k)$ are called generalized order statistics based on the distribution function F with density function f . The joint density function of the first *r* generalized order statistics $X(1,n,m,k),\ldots,X(r,n,m,k)$ is given by

$$f^{X(1,n,m,k),\ldots,X(r,n,m,k)}(x_{(1)},x_{(2)},\ldots,x_{(r)}) = C_{r-1}\left(\prod_{i=1}^{r-1}[1-F(x_{(i)})]^{m} f(x_{(i)})\right)[1-F(x_{(r)})]^{\gamma_r - 1} f(x_{(r)}) \qquad (1.5)$$

where

$$F^{-1}(0) < x_{(1)} \le x_{(2)} \le \ldots \le x_{(n)} < F^{-1}(1) \quad \text{and} \quad C_{r-1} = \prod_{j=1}^{r}\gamma_j \,, \;\; r = 1,2,\ldots,n, \;\; \gamma_n = k$$

in the case of $m = 0$ and $k = 1$, equation (1.5) reduces to the joint density of the first *r* ordinary order statistics. If $m = -1$, $k = 1$, then (1.5) will be the joint density of the first r upper record values. For more details about gOS's, see Kamps (1995).

Ahsanallah (1996), Studied the distribution properties of gOS's from a uniform distribution based on the first *n* gOS's. Habibullah and Ahsanullah (2000) obtained the estimators of the parameters of Pareto II distribution based on gOS's. Ahsanallah (2000), Studied some distributional properties of the gOS's from two parameter exponential distribution. Jaheen (2002), considered the prediction of future gOS's from a general class of distributions using Bayesian two-sample prediction technique. Jaheen (2005) estimated the parameters of the Burr type XII distribution based on gOS's and upper record using MLE, Bayesian and approximate Bayes due to Lindely (1980) methods. Malinowska et al. (2006), derived the minimum variance linear unbiased estimators and the maximum likelihood estimators for Burr XII model based on n-selected gOS's. Burkschat et al. (2007), evaluated the estimators of the parameters of a location-scale family containing generalized Pareto distribution based on several samples of gOS's. Aboeleneen (2010), discussed Bayesian and non-Bayesian estimation based on generalized order statistics from Weibull distribution ,he obtained the estimators of the parameters and confidence intervals for progressively censoring type II and record values.

In this paper, The maximum likelihood and the Bayesian methods are applied to estimate the unknown parameters of Weibull extension model by using the generalized order statistics. Addition, asymptotic variance-covariance matrix of the estimators is given. Simulation studies have been performed using computer software for illustrating the new results for estimation problem.

2. Maximum Likelihood Estimation

Suppose that $X(1,n,m,k),(2,n,m,k),.....,(r,n,m,k)$, $k > 0$ and $m \in R$ are the first *r* generalized ordered statistics from a sample of size *n* drawn from the Weibull extension population (1.1), then the likelihood function can be obtained from (1.1), (1.2) and (1.5), as follows

$$L(\lambda,\alpha,\beta) = C_{r-1}.\lambda^r \beta^r \left[\prod_{i=1}^{r}\left(\frac{x_i}{\alpha}\right)^{\beta-1}\right].\exp\left[\sum_{i=1}^{r}\left(\frac{x_i}{\alpha}\right)^{\beta}\right] \times \exp\left[\lambda\alpha\sum_{i=1}^{r}\left(1-e^{(x_i/\alpha)^{\beta}}\right)+\lambda\alpha m\sum_{i=1}^{r-1}\left(1-e^{(x_i/\alpha)^{\beta}}\right)\right]$$
$$\times \exp\left[\lambda\alpha(\gamma_r-1)\left(1-e^{(x_r/\alpha)^{\beta}}\right)\right] \tag{2.1}$$

where $\gamma_r = k + (n-r)(m+1)$ and $x_i = x_{(i)}$. It is usually easier to maximize the natural logarithm of the likelihood function rather than the likelihood function itself. Therefore, the logarithm of the likelihood function is

$$\ln L \propto r\ln\lambda + r\ln\beta + \lambda\alpha\Psi + (\beta-1)\sum_{i=1}^{r}\ln\left(\frac{x_i}{\alpha}\right) + \sum_{i=1}^{r}\left(\frac{x_i}{\alpha}\right)^{\beta} - \lambda\alpha\sum_{i=1}^{r}e^{(x_i/\alpha)^{\beta}} - \lambda\alpha m\sum_{i=1}^{r-1}e^{(x_i/\alpha)^{\beta}}$$
$$- \lambda\alpha.(\gamma_r - 1).e^{(x_r/\alpha)^{\beta}} \tag{2.2}$$

where $\Psi = \left(r + m(r-1) + (\gamma_r - 1)\right)$.

Maximum likelihood estimators $\hat{\alpha}, \hat{\lambda}$ and $\hat{\beta}$ are the solutions of the system of equations obtained by equating the first partial derivatives of the natural logarithm of the likelihood function with respect to α, λ and β to zero. The maximum likelihood estimator $\hat{\lambda}$ for λ can be shown to be the form

$$\hat{\lambda} = \frac{r}{\hat{\alpha}.\left[\mathrm{H} - \Psi\right]} \tag{2.3}$$

where $\mathrm{H} = \left(\sum_{i=1}^{r}e^{\hat{z}_i} + m\sum_{i=1}^{r-1}e^{\hat{z}_i} + (\gamma_r - 1).e^{\hat{z}_r}\right)$ and $z_i = \left(\frac{x_i}{\alpha}\right)^{\beta}$, $z_r = \left(\frac{x_r}{\alpha}\right)^{\beta}$.

the estimators $\hat{\beta}$ and $\hat{\alpha}$ for β and α respectively can be obtained as the solution of the following equations:

$$r + \sum_{i=1}^{r}\hat{z}_i \ln\hat{z}_i\left[1 + \hat{z}_i^{-1} - \hat{\lambda}\hat{\alpha}e^{\hat{z}_i}\right] - \hat{\lambda}\hat{\alpha}.m\sum_{i=1}^{r-1}e^{\hat{z}_i} z_i \ln\hat{z}_i - \hat{\lambda}\hat{\alpha}(\gamma_r - 1)e^{\hat{z}_r}\hat{z}_r\ln\hat{z}_r = 0,$$

and

$$\hat{\lambda}\left[\hat{\alpha}\Psi - r(\hat{\beta}-1) - \hat{\beta}\sum_{i=1}^{r}\hat{z}_i - \hat{\alpha}\sum_{i=1}^{r}e^{\hat{z}_i}(1-\hat{\beta}\hat{z}_i) - \hat{\alpha}m\sum_{i=1}^{r-1}e^{\hat{z}_i}(1-\hat{\beta}\hat{z}_i) - \hat{\alpha}(\gamma_r - 1)e^{\hat{z}_r}(1-\hat{\beta}\hat{z}_r)\right] = 0 \tag{2.4}$$

equations (2.3) and (2.4) can not be solved analytically, statistical software can be used to solve these equations numerically.

The logarithm of the likelihood function (2.2) can be used to construct Fisher information matrix $\boldsymbol{I}(\boldsymbol{\theta})$, the observed information matrix with respect to λ, β and α are obtained by replacing λ, β and α with $\hat{\lambda}$, $\hat{\beta}$ and $\hat{\alpha}$ respectively, the elements of the observed information matrix are as follows:

$$-\left.\frac{\partial^2 \ln L}{\partial\lambda^2}\right|_{\hat{\alpha},\hat{\beta},\hat{\lambda}} = \frac{r}{\hat{\lambda}^2}$$

$$-\frac{\partial^2 \ln L}{\partial \beta^2}\bigg|_{\hat\alpha,\hat\beta,\hat\lambda} = \frac{1}{\hat\beta^2}\Bigg[r - \sum_{i=1}^{r} \hat z_i .\ln^2 \hat z_i + \hat\lambda\hat\alpha \sum_{i=1}^{r} e^{\hat z_i} \hat z_i (1+\hat z_i)\ln^2 \hat z_i$$

$$+ \hat\lambda\hat\alpha m \sum_{i=1}^{r-1} e^{\hat z_i} \hat z_i (1+\hat z_i)\ln^2 \hat z_i + \hat\lambda\hat\alpha(\gamma_r - 1) e^{\hat z_r} \hat z_r (1+\hat z_r)\ln^2 \hat z_r \Bigg]$$

$$-\frac{\partial^2 \ln L}{\partial \alpha^2}\bigg|_{\hat\alpha,\hat\beta,\hat\lambda} = \frac{1}{\hat\alpha^2}\Bigg[r(1-\hat\beta) - \hat\beta^2 \sum_{i=1}^{r} \hat z_i - \hat\beta \sum_{i=1}^{r} \hat z_i + \hat\lambda\hat\alpha\hat\beta \sum_{i=1}^{r} e^{\hat z_i} \hat z_i \left\{\hat\beta \hat z_i + \hat\beta - 1\right\} + \hat\lambda\hat\alpha\hat\beta m \sum_{i=1}^{r-1} e^{\hat z_i} z_i \left\{\hat\beta \hat z_i + \hat\beta - 1\right\}$$

$$+ (\gamma_r - 1)\hat\lambda\hat\alpha\hat\beta . e^{\hat z_r} \hat z_r \left\{\hat\beta \hat z_r + \hat\beta - 1\right\} \Bigg]$$

$$-\frac{\partial^2 \ln L}{\partial \alpha \partial \beta}\bigg|_{\hat\alpha,\hat\beta,\hat\lambda} = \frac{r}{\hat\alpha} + \frac{1}{\hat\alpha} \sum_{i=1}^{r} \hat z_i (1+\ln \hat z_i) - \hat\lambda \sum_{i=1}^{r} e^{\hat z_i} \hat z_i \left[1 + \ln \hat z_i \left(1 + \hat z_i - \frac{1}{\hat\beta}\right)\right]$$

$$- \hat\lambda m \sum_{i=1}^{r-1} e^{\hat z_i} \hat z_i \left[1 + \ln \hat z_i \left(1 + \hat z_i - \frac{1}{\hat\beta}\right)\right]$$

$$- \hat\lambda(\gamma_r - 1). e^{\hat z_r} \hat z_r \left[1 + \ln \hat z_r \left(1 + \hat z_r - \frac{1}{\hat\beta}\right)\right]$$

$$-\frac{\partial^2 L}{\partial \alpha \partial \lambda}\bigg|_{\hat\alpha,\hat\beta,\hat\lambda} = -\Psi - \sum_{i=1}^{r} e^{\hat z_i} \left(1 - \hat\beta \hat z_i\right) + m \sum_{i=1}^{r-1} e^{\hat z_i} \left(1 - \hat\beta \hat z_i\right) + (\gamma_r - 1) e^{\hat z_r} \left(1 - \hat\beta \hat z_r\right),$$

and

$$-\frac{\partial^2 \ln L}{\partial \beta \partial \lambda}\bigg|_{\hat\alpha,\hat\beta,\hat\lambda} = \frac{\hat\alpha}{\hat\beta}\Bigg[\sum_{i=1}^{r} e^{\hat z_i} \hat z_i \ln \hat z_i + m \sum_{i=1}^{r-1} e^{\hat z_i} \hat z_i \ln \hat z_i + (\gamma_r - 1)\frac{\hat\alpha}{\hat\beta}. e^{\hat z_r} \hat z_r \ln \hat z_r \Bigg] \tag{2.5}$$

Again computer facilities and numerical techniques are needed to evaluate (2.5).

3. Bayesian Estimation

Bayesian method is used to obtain the estimators of the unknown parameters of the WEM, Bayesian estimators and Bayesian risk are also obtained using the symmetric squared error loss. Elshahat (2007-a) assumed a non-informative independent prior distributions for the parameters α, λ and β as

$$g_1(\alpha) \propto \frac{1}{q}, \qquad 0 < \alpha < q,$$

$$g_2(\lambda) \propto \frac{1}{\lambda}, \qquad \lambda > 0, \tag{3.1}$$

and

$$g_3(\beta) \propto \frac{1}{a}, \qquad 0 < \beta < a$$

Therefore the joint prior for α, λ and β is given by

$$g(\alpha, \lambda, \beta) \propto \frac{1}{a\, q\, \lambda}, \; 0 < \alpha < q, \; \lambda > 0, \; 0 < \beta < a \tag{3.2}$$

Combining the likelihood function (2.1) with (3.2), the trivariate posterior density of α, λ and β is given by

$$q(\alpha, \lambda, \beta \mid x) = \frac{\lambda^{r-1}\beta^{r}\left[\prod_{i=1}^{r} w_i^{\beta-1}\right] e^{\sum_{i=1}^{r} z_i + \lambda\alpha \sum_{i=1}^{r}\left(1-e^{z_i}\right) + \lambda\alpha m \sum_{i=1}^{r-1}\left(1-e^{z_i}\right)} \; e^{\lambda\alpha(\gamma_r - 1)\left(1-e^{z_r}\right)}}{T} \tag{3.3}$$

Where T is the normalized constant and is given by

$$T = \int_0^a \int_0^\infty \int_0^q \lambda^{r-1}\beta^{r}\left[\prod_{i=1}^{r} w_i^{\beta-1}\right] e^{\sum_{i=1}^{r} z_i + \lambda\alpha \sum_{i=1}^{r}\left(1-e^{z_i}\right) + \lambda\alpha m \sum_{i=1}^{r-1}\left(1-e^{z_i}\right)} \; e^{\lambda\alpha(\gamma_r - 1)\left(1-e^{z_r}\right)} . d\alpha . d\lambda . d\beta$$

The marginal posteriors of α, λ and β are obtained by integrating the joint posterior distribution (3.3) with respect to other parameters, that is the posterior density of α is

$$q_1(\alpha \mid x) = \frac{\int_0^a \int_0^\infty \lambda^{r-1}\beta^{r}\left[\prod_{i=1}^{r} w_i^{\beta-1}\right] e^{\sum_{i=1}^{r} z_i + \lambda\alpha \sum_{i=1}^{r}\left(1-e^{z_i}\right) + \lambda\alpha m \sum_{i=1}^{r-1}\left(1-e^{z_i}\right)} \; e^{\lambda\alpha(\gamma_r - 1)\left(1-e^{z_r}\right)} . d\lambda . d\beta}{T}, \quad 0 < \alpha < q \tag{3.4}$$

Similarly, the posterior density of λ is

$$q_2(\lambda \mid x) = \frac{\lambda^{r-1}\int_0^a \int_0^q \beta^{r}\left[\prod_{i=1}^{r} w_i^{\beta-1}\right] e^{\sum_{i=1}^{r} z_i + \lambda\alpha \sum_{i=1}^{r}\left(1-e^{z_i}\right) + \lambda\alpha m \sum_{i=1}^{r-1}\left(1-e^{z_i}\right)} \; e^{\lambda\alpha(\gamma_r - 1)\left(1-e^{z_r}\right)} . d\alpha . d\beta}{T}, \quad \lambda > 0 \tag{3.5}$$

Finally, the posterior density of β is

$$q_3(\beta \mid x) = \frac{\beta^{r}\int_0^\infty \int_0^q \lambda^{r-1}\left[\prod_{i=1}^{r} w_i^{\beta-1}\right] e^{\sum_{i=1}^{r} z_i + \lambda\alpha \sum_{i=1}^{r}\left(1-e^{z_i}\right) + \lambda\alpha m \sum_{i=1}^{r-1}\left(1-e^{z_i}\right)} \; e^{\lambda\alpha(\gamma_r - 1)\left(1-e^{z_r}\right)} . d\alpha . d\lambda}{T} \quad 0 < \beta < a \tag{3.6}$$

Using squared error loss function and equations (3.4) to (3.6) Bayes estimators can be obtained as the posterior mean as follows

$$\tilde{\alpha} = \left[\int_0^q \alpha .\, q_1(\alpha \mid x).d\,\alpha \right],$$

$$\tilde{\lambda} = \left[\int_0^\infty \lambda . q_2(\lambda \mid x).d\,\lambda \right],$$

and

$$\tilde{\beta} = \left[\int_0^a \beta . q_3(\beta \mid x).d\,\beta \right]$$

Also, Bayes risk can be obtained as the posterior variance. Equations (3-4) to (3-6) are very hard to solve. An iterative procedure is needed to solve these equations numerically using statistical package.

4. Simulation Results

Simulation studies have been performed using MATHCAD for illustrating the new results for estimation problem. 5000 random samples of sizes 10, 20 and 30 were generated from Weibull extension model and used to illustrate the properties of maximum likelihood estimators, while 100 random samples of sizes 10, 30, 50 and 70 were generated from Weibull extension model to illustrate the properties of Bayesian estimators.

MATHCAD package is used to evaluate the ML estimators under generalized order statistics using equations (2.3) and (2.4) for $(k = 1.5 \; and \; m = 0.5)$ and $(k = 1 \; and \; m = 1)$, and for different values of the parameters $(\alpha = 2.5, \beta = 0.8, \lambda = 0.1)$. The performance of the resulting estimates of the parameters has been considered in terms of their square root of the mean square error ($\sqrt{\text{MSE}}$). Furthermore, for each estimators the Pearson type of distributions will be obtained. The steps of the simulation procedure will be as follows

Step 1: 5000 random samples of sizes 10, 20 and 30 were generated from Weibull extension model. If U has a uniform (0,1) random number, then

$$x = \alpha \left[\ln\left(1 - \ln(1-u) / \lambda\alpha\right) \right]^{1/\beta}$$

follows the Weibull extension model. The true parameters values are selected for all combinations for the following values $\alpha = 2.5, \beta = 0.8, \lambda = 0.1$.

Step 2: Choose the number of failure r to be less than the sample size n and to be equal the sample size.

Step 3: Newton-Raphson method was used for solving the equations (2.3) and (2.4), respectively, to obtain the ML estimates of the unknown parameters α, β and λ.

Step 4: For 5000 estimates the square root of the MSE, and the moments about the mean are obtained to compute the skewness, kurtosis and Pearson criterion K_P to determine Pearson type of the estimators.

Simulation results are displayed in Tables 1 and 2. Table 1 gives the ML estimates of the unknown parameters, the square root of the MSE and Pearson type distribution of the estimators for $k = 1.5$ *and* $m = 0.5$. While Table 2 present ML estimates of the unknown parameters, the square root of the MSE and Pearson type distribution of estimators for $k = 1$ *and* $m = 1$. From these tables, we conclude that:

1. As the sample size increases $\sqrt{\text{MSE}}$ of the estimated parameters decrease. This indicates that the maximum likelihood estimates tend to its true parameters values.
2. Generally, we observe that the estimator for the unknown parameter α has Pearson type I distribution. Also, we observe that the estimators for the unknown parameters β and λ have Pearson type IV distribution.

Also, MATHCAD package is used to evaluate Bayes estimators under generalized order statistics using equations (3.4) to (3.6) for $k = 1$ *and* $m = 1$, and for parameters values $\alpha = 5$, $\beta = 0.8, \lambda = 0.2$. The performance of the resulting estimates of the parameters has been considered in terms of the mean square error (MSE). The Simulation procedures will described below:

Step 1: 100 random samples of sizes 10, 30, 50 and 70 were generated from Weibull extension model. The selected values for parameters are $\alpha = 5$, $\beta = 0.8, \lambda = 0.2$.

Step 2: Choose the number of failure r to be less than the sample size n and to be equal the sample size.

Step 3: Numerical integration method was used for solving the equations (3.4)-(3.6), respectively to obtain 100 Bayes estimates under squared error loss function as the posterior mean and the mean square error (MSE).

Simulation results are displayed in Table 3, which gives the posterior mean and MSE. Simulation studies are adopted for different sized samples. It is noted that the mean square error is decreasing when n is increasing. Also, we observe that the value of MSE in the case of $r = n$ is smaller than its value in the case of $r < n$.

Table (1): Maximum likelihood estimates, square root of MSE and Pearson type of distributions for the estimates of the unknown parameters α, β and λ, for different samples size under generalized order statistics for $k = 1$ and $m = 1$ for N = 5000 repetitions.

		Parameter			ML estimate			Pearson Type		
n	r	α	β	λ	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
10	8	**2.5**	**0.8**	**0.1**	2.05	0.72	0.08	VI	I	IV
		Square root of MSE			0.84	0.27	0.04			
10	10	**2.5**	**0.8**	**0.1**	2.05	0.73	0.09	VI	I	IV
		Square root of MSE			0.82	0.24	0.03			
20	15	**2.5**	**0.8**	**0.1**	2.25	0.67	0.08	I	I	IV
		Square root of MSE			3.98	0.21	0.03			
20	20	**2.5**	**0.8**	**0.1**	2.29	0.78	0.07	I	I	VI
		Square root of MSE			0.64	0.19	0.04			
30	20	**2.5**	**0.8**	**0.1**	1.92	0.71	0.06	I	IV	VI
		Square root of MSE			0.71	0.17	0.74			
30	30	**2.5**	**0.8**	**0.1**	2.06	0.78	0.06	I	I	I
		Square root of MSE			0.61	0.13	0.05			

Table (2): Maximum likelihood estimates, square root of MSE and Pearson type of distributions for the estimates of the unknown parameters α, β and λ, for different samples size under generalized order statistics for $k = 1.5$ and $m = 0.5$ for N = 5000 repetitions

n	*r*	Parameter			ML estimate			Pearson Type		
10	8	**2.5**	**0.8**	**0.1**	2.12	0.77	0.08	I	VI	VI
		Square root of MSE			1.61	0.39	0.04			
10	10	**2.5**	**0.8**	**0.1**	2.31	0.79	0.08	VI	I	VI
		Square root of MSE			0.79	0.22	0.04			
20	15	**2.5**	**0.8**	**0.1**	2.2	0.77	0.08	IV	I	VI
		Square root of MSE			0.6	0.21	0.03			
20	20	**2.5**	**0.8**	**0.1**	2.15	0.77	0.08	IV	VI	I
		Square root of MSE			0.59	0.15	0.04			
30	20	**2.5**	**0.8**	**0.1**	2.036	0.749	0.08	I	VI	I
		Square root of MSE			0.62	0.18	0.04			
30	30	**2.5**	**0.8**	**0.1**	2.01	0.701	0.08	I	I	IV
		Square root of MSE			0.67	0.15	0.03			

Table (3): Bayesian estimates and MSE under squared error loss function for different samples sizes under generalized order statistics with m = 1 and k = 1 and for $\alpha = 5, \beta = 0.8$ and $\lambda = 0.2$

n* = *r	Parameters	Bayes Estimate (Posterior Mean)	MSE	*r*	Bayes Estimate (Posterior Mean)	MSE
10	α	5.625	0.974	8	4.274	8.2816
	β	0.845	0.185		0.719	0.182
	λ	0.212	0.0006		0.158	0.0033
30	α	5.239	0.0571	20	5.638	0.407
	β	0.821	0.173		0.739	0.164
	λ	0.198	0.00001		0.211	0.0001
50	α	4.857	0.0204	35	4.848	0.023
	β	0.809	0.159		0.822	0.16
	λ	0.203	0.00001		0.196	0.00002
70	α	5.005	0.00003	50	4.995	0
	β	0.798	0.1		0.801	0.11
	λ	0.201	0.0000006		0.202	0

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