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Stability of Quadratic Functional Equation in RN-Spaces

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Abstract

In this paper, using direct method, we prove the generalized Hyers-Ulam stability of the following quadratic functional equation

$$\sum_{1\le i<j\le m} f(x_i+x_j)+f(x_i-x_j)=2(m-1)\sum_{i=1}^{m} f(x_i)$$

for all $x_1, x_2, \cdots, x_m \in X$, where $m \ge 2$ in random normed spaces.

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1 Introduction

A classical question in the theory of functional equations is the following: "When is it true that a function which approximately satisfies a functional equation D must be close to an exact solution of D?.

If the problem accepts a solution, we say that the equation D is stable. The first stability problem concerning group homomorphisms was raised by Ulam [8] in 1940.

We are given a group G and a metric group G' with metric $d(.,.)$. Given $\varepsilon > 0$, dose there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $d(f(xy), f(x)f(y)) < \delta$, for all $x, y \in G$, then a homomorphism $h : G \to G$ exists with $d(f(x), h(x)) < \varepsilon$ for all $x \in G$?.

Ulam's problem was partially solved by Hyers [4] in 1941. In 1978, Th. M. Rassias [5] formulated and proved the following theorem, which implies Hyers's Theorem as a special case. Suppose that E and F are real normed spaces with F a complete normed space, $f : E \to F$ is a mapping such that for each fixed $x \in E$ the mapping $t \to f(tx)$ is continuous on R, and let there exist $\varepsilon > 0$ and $p \in [0, 1)$ such that for all $x, y \in E$

$$\frac{||f(x+y) - f(x) - f(y)||}{||x||^p + ||y||^p} \leq \varepsilon \tag{1}$$

Then there exists a unique linear mapping $T : E \to F$ such that such that for all $x \in E$

$$||f(x) - T(x)|| \leq \frac{\varepsilon||x||^p}{1 - 2^{p-1}} \tag{2}$$

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{3}$$

is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [7] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [1] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. In [2], Czerwik proved the generalized Hyers-Ulam stability of the quadratic functional equation.

2 Preliminary Notes

In the sequel, we shall adopt the usual terminologies, notions and conventions of the theory of random normed spaces(see [6]). Throughout this paper, the space of all probability distribution functions is denoted by Λ^+. Elements of

Λ^+ are functions $F : R \cup [-\infty, +\infty] \to [0,1]$, such that F is left continuous and nondecreasing on R and $F(0) = 0, F(+\infty) = 1$. It's clear that the subset

$$D^+ = \{F \in \Lambda^+ : l^- F(-\infty) = 1\},$$

where $l^- f(x) = \lim_{t \to x^-} f(t)$, is a subset of Λ^+. The space Λ^+ is partially ordered by the usual point-wise ordering of functions, that is for all $t \in R$, $F \le G$ if and only if $F(t) \le G(t)$.

Definition 2.1 *A function $T : [0,1]^2 \to [0,1]$ is a continuous triangular norm (briefly a t-norm) if T satisfies the following conditions:*
(i) T is commutative and associative;
(ii) T is continuous;
(iii) $T(x,1) = x$ for all $x \in [0,1]$;
(iv) $T(x,y) \le T(z,w)$ whenever $x \le z$ and $y \le w$ for all $x,y,z,w \in [0,1]$.

Definition 2.2 *A random normed space(briefly RN-space) is a triple (X, μ, T), where X is a vector space, T is a continuous t-norm and $\mu : X \to D^+$ is a mapping such that the following conditions hold:*
(i) $\mu_x(t) = H_0(t)$ for all $t > 0$ if and only if $x = 0$;
(ii) $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $\alpha \in R$, $\alpha \neq 0$, $x \in X$ and $t \ge 0$.
(iii)$\mu_{x+y}(t+s) \ge T(\mu_x(t), \mu_y(s))$, for all $x, y \in X$ and $t, s \ge 0$.

Definition 2.3 *Let (X, μ, T) be an RN-space. A sequence $\{x_n\}$ in X is said to be converges to $x \in X$ if for all $t > 0$, $lim_{n\to\infty} \mu_{x_n - x}(t) = 1$.*

Definition 2.4 *A sequence $\{x_n\}$ in (X, μ, T) is said to be a Cauchy sequence in X if for all $t > 0$, $lim_{n\to\infty} \mu_{x_n - x_m}(t) = 1$. The RN-space (X, μ, T) is said to be complete if every Cauchy sequence in X is convergent.*

Theorem 2.5 *If (X, μ, T) is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$.*

3 Main Results

Lemma 3.1 *Let X and Y be vector spaces. A mapping $f : X \to Y$ satisfies the functional equation*

$$\sum_{1 \le i < j \le m} f(x_i \pm x_j) = 2(m-1) \sum_{i=1}^{m} f(x_i) \tag{4}$$

if and only if f is quadratic.

Proof: Let f be a quadratic function. Assume the equation (4) is true for n by induction argument. By (3)

$$f(x_i + x_{n+1}) + f(x_i - x_{n+1}) - 2f(x_i) - 2f(x_{n+1}) = 0 \tag{5}$$

for all $i =, \cdots, n$. Adding up (3) and (5), we have the desired equation (3) for $n+1$. Conversely, let f satisfy the equation (3). By letting $x_i = 0$ for all $i = 1, 2, ..., n$, we have $f(0) = 0$. Replacing $x_i = 0$ for all $i = 3, 4, ..., n$, we obtain the equation

$$f(x_1 + x_2) + f(x_1 - x_2) = 2f(x_1) + 2f(x_2) = 0 \tag{6}$$

which implies that f is quadratic. The proof is complete.

Theorem 3.2 *Let X be a real linear space, (Z, μ', min) be an RN-space, $\phi : X^m \to Z$ be a function such that for some $0 < \alpha < 4$,*

$$\mu'_{\phi(2x_1,...,2x_m)}(t) \geq \mu'_{\alpha\phi(x_1,...,x_m)}(t) \quad \forall x_1, ..., x_n \in X,\ t > 0 \tag{7}$$

$f(0) = 0$ and for all $x_1, \cdots, x_m \in X$ and $t > 0$, $\lim_{n\to\infty} \mu'_{\phi(2^n x_1,...,2^n x_n)}(4^n t) = 1$. Let (Y, μ, min) be a complete RN-space. If $f : X \to Y$ is a mapping such that for all $x_1, ..., x_m \in X$ and $t > 0$

$$\mu_{\sum_{1\leq i<j\leq m} f(x_i+x_j)+f(x_i-x_j)-2\sum_{i=1}^{m} f(x_i)}(t) \geq \mu'_{\phi(x_1,...,x_m)}(t), \tag{8}$$

then the limit $Q(x) := \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$ exist for all $x \in X$ and $Q : X \to Y$ is a unique quadratic mapping satisfying the inequality

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\phi(x,x,\cdots,x)}\Big(\frac{m(m-1)(4-\alpha)t}{2}\Big). \tag{9}$$

Proof: Putting $x_1 = x_2 = \cdots = x_m = x$ in (8), we obtain

$$\mu_{\frac{m(m-1)}{2} f(2x) - 2m(m-1)f(x)}(t) \geq \mu'_{\phi(x,x,\cdots,x)}(t). \tag{10}$$

So

$$\mu_{\frac{f(2x)}{4} - f(x)}(t) \geq \mu'_{\phi(x,x,\cdots,x)}(2m(m-1)t). \tag{11}$$

Replacing x by $2^n x$ in (11) and using (7), we obtain

$$\begin{aligned} \mu_{\frac{f(2^{n+1}x)}{4^{n+1}} - \frac{f(2^n x)}{4^n}}(t) &\geq \mu'_{\phi(2^n x, 2^n x, \cdots, 2^n x)}(2 \times 4^n m(m-1)t) \qquad (12)\\ &\geq \mu'_{\phi(x,x,\cdots,x)}\Big(\frac{2 \times 4^n m(m-1)t}{\alpha^n}\Big). \end{aligned}$$

So by (12), we obtain

$$\begin{aligned}
\mu_{\frac{f(2^{n+1}x)}{4^{n+1}}-f(x)}\Big(\sum_{k=0}^{n-1}\frac{t\alpha^k}{2\times 4^k m(m-1)}\Big) &= \mu_{\sum_{k=0}^{n-1}\frac{f(2^{k+1}x)}{4^{k+1}}-\frac{f(2^k x)}{4^k}}\Big(\sum_{k=0}^{n-1}\frac{t\alpha^k}{2\times 4^k m(m-1)}\Big)\\
&\geq T_{k=0}^{n-1}\Big(\mu_{\frac{f(2^{k+1}x)}{4^{k+1}}-\frac{f(2^k x)}{4^k}}\Big(\frac{t\alpha^k}{2\times 4^k m(m-1)}\Big)\Big)\\
&\geq T_{k=0}^{n-1}(\mu'_{\phi(x,x,\cdots,x)}(t)) \qquad (13)\\
&= \mu'_{\phi(x,x,\cdots,x)}(t).
\end{aligned}$$

This implies that

$$\mu_{\frac{f(2^n x)}{4^n}-f(x)}(t) \geq \mu'_{\phi(x,x,\cdots,x)}\Big(\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{2\times 4^k m(m-1)}}\Big). \qquad (14)$$

Replacing x by $2^p x$ in (14), we obtain

$$\mu_{\frac{f(2^{p+n}x)}{4^{n+p}}-\frac{f(2^p x)}{4^p}}(t) \;\geq\; \mu'_{\phi(x,x,\cdots,x)}\Big(\frac{t}{\sum_{k=p}^{p+n-1}\frac{\alpha^k}{2\times 4^k m(m-1)}}\Big). \qquad (15)$$

As $\lim_{p,n\to\infty}\mu'_{\phi(x,x,\cdots,x)}\Big(\frac{t}{\sum_{k=p}^{p+n-1}\frac{\alpha^k}{2\times 4^k m(m-1)}}\Big) = 1$ then $\Big\{\frac{f(2^n x)}{4^n}\Big\}_{n=1}^{+\infty}$ is a Cauchy sequence in complete RN-space (Y,μ,min), so there exist some point $C(x)\in Y$ such that $\lim_{n\to\infty}\frac{f(2^n x)}{4^n} = Q(x)$. Fix $x\in X$ and put $p=0$ in (15). Then we obtain

$$\mu_{\frac{f(2^n x)}{4^n}-f(x)}(t) \geq \mu'_{\phi(x,x,\cdots,x)}\Big(\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{2\times 4^k m(m-1)}}\Big). \qquad (16)$$

and so, for every $\epsilon>0$, we have

$$\begin{aligned}
\mu_{C(x)-f(x)}(t+\epsilon) &\geq T\Big(\mu_{Q(x)-\frac{f(2^n x)}{4^n}}(\epsilon), \mu_{\frac{f(2^n x)}{4^n}-f(x)}(t)\Big)\\
&\geq T\Big(\mu_{Q(x)-\frac{f(2^n x)}{4^n}}(\epsilon), \mu'_{\phi(x,x,\cdots,x)}\Big(\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{2\times 4^k m(m-1)}}\Big)\Big). \qquad (17)
\end{aligned}$$

Taking the limit as $n\to\infty$, we get

$$\mu_{C(x)-f(x)}(t+\epsilon) \geq \mu'_{\phi(x,x,\cdots,x)}\Big(\frac{m(m-1)(4-\alpha)t}{2}\Big). \qquad (18)$$

Since ϵ was arbitrary by taking $\epsilon\to 0$ in (18), we obtain

$$\mu_{C(x)-f(x)}(t) \geq \mu'_{\phi(x,x,\cdots,x)}\Big(\frac{m(m-1)(4-\alpha)t}{2}\Big). \qquad (19)$$

Replacing x and y by $2^n x$ and $2^n y$, respectively, in (8) and using this fact that $\lim_{n\to\infty}\mu'_{\phi(2^n x_1,\ldots,2^n x_n)}(4^n t) = 1$, we get for all $x_1,\cdots,x_n\in X$ and for all $t>0$

$$\sum_{1\leq i<j\leq m} Q(x_i+x_j) + Q(x_i-x_j) = 2\sum_{i=1}^{m} Q(x_i).$$

Therefore, the mapping Q is quadratic.
To prove the uniqueness of mapping Q, assume that there exist another additive mapping $R : X \to Y$ which satisfies (9). Since for all $n \in N$ and every $x \in X$, $Q(2^n x) = 4^n Q(x)$ and $R(2^n x) = 4^n R(x)$, we find that

$$\mu_{Q(x)-R(x)}(t) = \lim_{n\to\infty} \mu_{\frac{Q(2^n x)}{4^n} - \frac{R(2^n x)}{4^n}}(t). \tag{20}$$

So

$$\begin{aligned} \mu_{\frac{Q(2^n x)}{4^n} - \frac{R(2^n x)}{4^n}}(t) &\geq min\Big\{\mu_{\frac{Q(2^n x)}{4^n} - \frac{f(2^n x)}{4^n}}\Big(\frac{t}{2}\Big), \mu_{\frac{f(2^n x)}{4^n} - \frac{R(2^n x)}{4^n}}\Big(\frac{t}{2}\Big)\Big\} \\ &\geq \mu'_{\phi(2^n x, 2^n x, \cdots, 2^n x)}\Big(\frac{m(m-1)4^n(4-\alpha)t}{2}\Big) \\ &\geq \mu'_{\phi(x,x,\cdots,x)}\Big(\frac{m(m-1)4^n(4-\alpha)t}{2\alpha^n}\Big). \end{aligned} \tag{21}$$

Since $\lim_{p\to\infty} \frac{m(m-1)4^n(4-\alpha)}{2\alpha^n} = \infty$, we get

$$\lim_{p\to\infty} \mu'_{\phi(x,x,\cdots,x)} \frac{m(m-1)4^n(4-\alpha)t}{2\alpha^n} = 1.$$

Therefore, for all $t > 0$, $\mu_{Q(x)-R(x)}(t) = 1$ and so $Q(x) = R(x)$. This completes the proof.

Corollary 3.3 *Let X be a real linear space, (Z, μ', min) be an RN-space, and (Y, μ, min) a complete RN-spaces. Let $p \in (0, 1)$ and $z_0 \in Z$. If $f : X \to Y$ is a mapping with $f(0) = 0$ and for all $x_1, ..., x_m \in X$ and $t > 0$*

$$\mu_{\sum_{1\leq i<j\leq m} f(x_i+x_j)+f(x_i-x_j)-2\sum_{i=1}^m f(x_i)}(t) \geq \mu'_{(\sum_{i=1}^m ||x_i||^p)z_0}(t), \tag{22}$$

then there is a unique quadratic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{||x||^p z_0}\Big(\frac{m(m-1)(4-4^p)t}{2m}\Big). \tag{23}$$

Proof: Let $\alpha = 4^p$ and $\phi : X^m \to Z$ defined by $\phi(x_1, ..., x_m) = (\sum_{i=1}^m ||x_i||^p)z_0$. Applying Theorem (3.2), we get desired result.

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